Topology & Ultracold Atoms

Dear students,

The following pages contain the hand-written notes that I used for my three lectures at Les Houches [10-12 October 2018].

Please don't hesitate to contact me, in case something [i.e. a definition, a reference, a derivation...] is unclear.

Thanks again for attending the lectures, and for asking so many (good) questions!

Nathan

Brussels, October 22nd 2018.

Topology & Ultracold atoms

I]. Brief introduction to geometry and topology (in quantum physics)

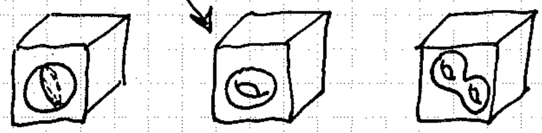
1°). Geometry versus topology

Take an object

⇝ 2 views a]. "Overall shape"

b]. Local details (curvature)

a]. "Overall shape" ⇒ topology : classifies geometric objects in terms of global properties



...

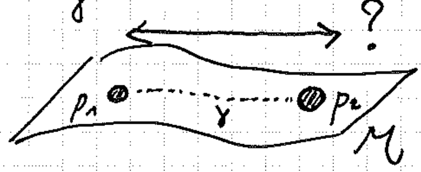
$\nu = 0$ $\nu = 1$ $\nu = 2$ ⇒ "genus" of surface

ν = topological invariant ("constant under smooth perturbations")

Classification: "two objects belong to the same class ⟺ they can be deformed one into another."

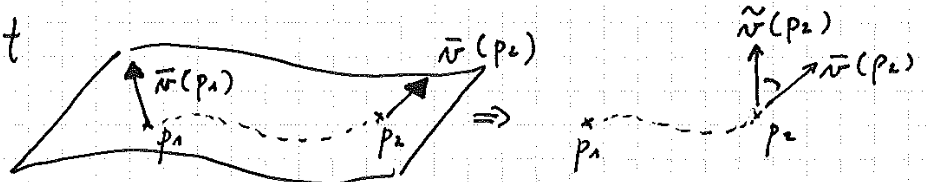
b]. Local details ⇒ geometry

* notion of distance

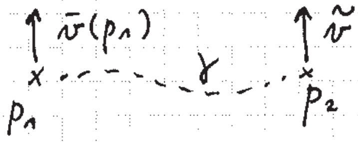


⇒ metric $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$

* notion of transport



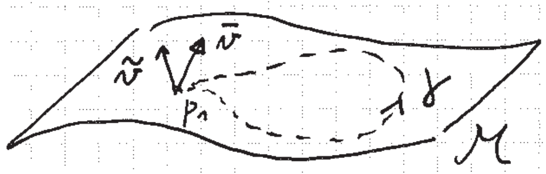
(allows one to define a derivative of vector fields over M).



$\tilde{v}(p_1)$: parallel transport of $\bar{v}$ along the curve γ

[this is realized through a "connection"].

* notion of mismatch under parallel transport



$\bar{v} \neq \tilde{v}$ after a loop γ

↳ signals (defines) a non-zero local curvature Ω

c]. Connecting geometry and topology

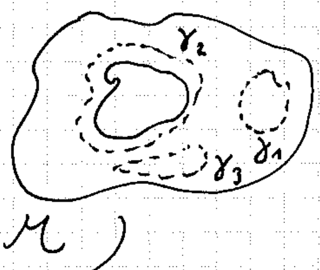
Gauss-Bonnet theorem (here for surfaces without edges)

$$\frac{1}{2\pi} \int_{\mathcal{M}} \underbrace{\Omega}_{\text{local curvature}} \, dS = \underbrace{2(1-g)}_{\text{top. invariant (genus)}}$$

↳ can be generalized to other manifolds ...

d]. Homotopy class

Consider a manifold $\mathcal{M}$ and all possible loops in $\mathcal{M}$ $\;(\gamma: I \to \mathcal{M})$

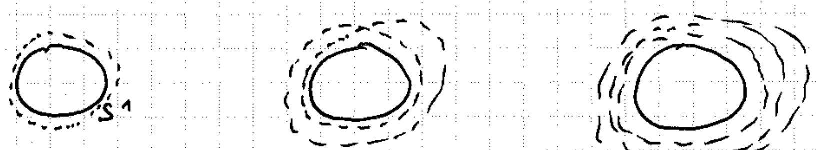


We introduce an equivalence relation for loops: "two loops γ_1 and γ_2 are equivalent (homotopic) if they can be continuously deformed one into another".

$\underbrace{\gamma_1 \equiv \gamma_3}_{[\alpha_0]} \not\equiv \underbrace{\gamma_2}_{[\alpha_1]}$: each class is characterized by an integer $n \in \mathbb{Z}$ (how many times the loop encircles the hole): $[\alpha_n]$

The set of classes (here $\mathbb{Z}$) is called "fundamental" or "first homotopy group" of $\mathcal{M}$. It is a top. invariant characterizing $\mathcal{M}$.

Ex$_1$: $\pi_1(S^1) = \mathbb{Z}$



$n=0$ $\quad$ $n=1$ (winding) $\quad$ $n=2$ (winding) $\quad$ ...

Ex$_2$: $\pi_1(\mathbb{T}) = \mathbb{Z} \oplus \mathbb{Z}$

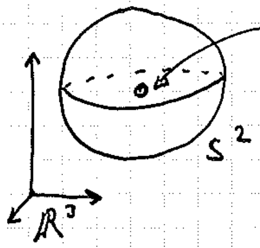
2 types of winding.

physics $\Rightarrow$ used to characterize topol. defects (monopoles, vortices, domain walls)

2°). Application in electromagnetism: From Dirac's monopole to the Chern number

Maxwell's equation $\bar{\nabla} \cdot \bar{B} = 0$ (absence of a magnetic monopole)

Dirac 1931: Let's add it $\bar{\nabla} \cdot \bar{B} = 4\pi \rho_m = 4\pi g \underbrace{\delta^3(\bar{r})}_{\text{point-like charge}}$



Exercise: Show $\bar{B} = g \dfrac{\bar{r}}{r^3}$ (radial field)

Let's calculate the flux through a sphere S^2:

$$\Phi = \oint_{S^2} \bar{B} \cdot d\bar{S} \overset{\text{div.}}{=} \int_V (\bar{\nabla} \cdot \bar{B})\, dV = 4\pi g \quad (\text{finite}).$$

⊙ Problem appear when introducing a gauge potential $\bar{A}$:

Write $\bar{B} = \bar{\nabla} \times \bar{A}$ and calculate the same flux:

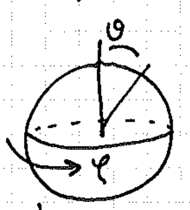
$$\Phi = \oint_{S^2} \bar{B} \cdot d\bar{S} = \oint_{S^2} (\bar{\nabla} \times \bar{A}) \cdot d\bar{S}$$

$$\overset{\text{div.}}{=} \int_V \underbrace{\bar{\nabla} \cdot (\bar{\nabla} \times \bar{A})}_{=0} \cdot d\bar{S} = 0$$

$\Rightarrow$ contradiction $[\,\Phi = 4\pi g\,]$!

What went wrong? We have assumed that $\bar{A}$ is globally defined over the entire space! WRONG!

Let's be concrete: write $\bar{A} = g(1 - \cos\theta)\, \bar{\nabla}\varphi$



equator: $\theta = \frac{\pi}{2}$

Exercise: show $\bar{\nabla} \times \bar{A} = \bar{B} = g\, \bar{r}/r^3$.

However: $\bar{A}$ is defined everywhere except at the $\boxed{\theta = \pi}$

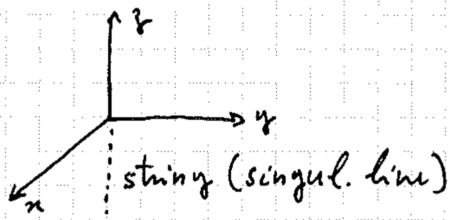
Note : $\bar{A} = \underbrace{\frac{1}{2r\sin\theta}\,(1-\cos\theta)}\,\underbrace{\bar{1}_\varphi}_{\text{not well defined at } \theta=0 \text{ and } \pi}$

$= 0$ for $\theta = 0$ ($\checkmark$)

$= \infty$ for $\theta = \pi$ ($\times$)

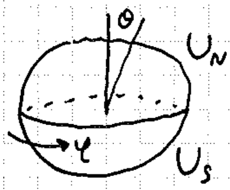
$\Rightarrow$ notion of Dirac string :

$\bar{A}$ is well defined except along the string (avoid !)



The approach of Wu & Yang [PRD 1975]

Let's use 2 local gauge potentials !



$$\bar{A}_N = \bar{A} = g(1-\cos\theta)\,\bar{\nabla}\varphi \qquad [\checkmark \text{ for } \theta \neq \pi]$$

$$\bar{A}_S = -g\,(1+\cos\theta)\,\bar{\nabla}\varphi \qquad [\checkmark \text{ for } \theta \neq 0]$$

We can recalculate the flux :

$$\Phi = \oint_{S^2} \bar{B}\cdot d\bar{S} = \int_{U_N} (\bar{\nabla}\times\bar{A}_N)\cdot d\bar{S} + \int_{U_S} (\bar{\nabla}\times\bar{A}_S)\cdot d\bar{S}$$

⚠ $U_{N,S}$ are not closed (have an edge $\equiv$ equator $\mathcal{E}$)

$$\overset{\text{Stokes}}{=} \int_{\mathcal{E}} (\bar{A}_N - \bar{A}_S)\cdot d\bar{\ell} = 2g\underbrace{\int_{\mathcal{E}} (\bar{\nabla}\varphi)\cdot d\bar{\ell}}_{=2\pi} = 4\pi g \quad \checkmark$$

* Quantization :

Consider the gauge transformation $\bar{A}_N - \bar{A}_S = \bar{\nabla}(2g\varphi)$

↳ this should be well defined at the "gluing" region $\mathcal{E}$ ($\theta = \pi/2$)

In terms of the wave functions (in different gauges), we have : $\Psi_S(\bar{r}) = \Psi_N(\bar{r})\,\exp[-i\,2g\varphi]$ $\quad (e=\hbar=c=1)$

Reminder : $\begin{cases} \bar{A} \to \tilde{A} = \bar{A} + \bar{\nabla}\chi \\ \Psi(\bar{r}) \to \tilde{\Psi}(\bar{r}) = \Psi(\bar{r})\, e^{i\frac{e}{\hbar c}\chi(\bar{r})} \end{cases}$

Hence, we have $\Psi_s(\bar{r}) = \Psi_N(\bar{r}) \exp\left[-i\, 2 g\, \varphi\right]$ at $\theta = \pi/2$

and we require that $\Psi_{N,S}(\bar{r})$ are single-valued over $\mathcal{E}$.

$$\Longrightarrow \Psi_{S,N}(\varphi = 0) = \Psi_{S,N}(\varphi = 2\pi^-)$$

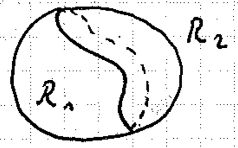
$$\Longleftrightarrow \boxed{2 g \in \mathbb{Z}}$$

Reintroducing $e, \hbar, c$ $\Longrightarrow$ $\dfrac{2 e g}{\hbar c} \in \mathbb{Z}$: Dirac quantization

"if there's a monopole in the Universe $\Longrightarrow$ e is quantized".

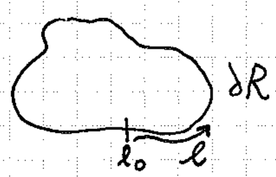
Link with topology 1 :

Let us be general and introduce 2 regular potentials over S^2:



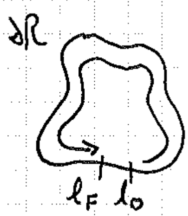
$\bar{A}_1$ well defined in R_1
$\bar{A}_2$ well defined in R_2 (+ boundary ∂R)

Suppose there's a single boundary ∂R :



on ∂R : $\begin{cases} \bar{A}_2 = \bar{A}_1 + \bar{\nabla}\chi \\ \Psi_2(l) = \Psi_1(l)\, e^{i\chi(l)} \end{cases}$

Single-valuedness of $\Psi_{1,2}(l) \Longrightarrow e^{i\chi}$: loop $\partial R \longrightarrow U(1)$



$\left[\chi(l_F) - \chi(l_0) = 2\pi \times n\right]$ with $n \in \mathbb{Z}$

Here, "n" counts how many times the loop $e^{i\chi}$ winds around $U(1) \equiv S^1$.

$\Rightarrow$ "n" sets the homotopy class (top. inv.)

Besides, we have: (Stokes theorem)

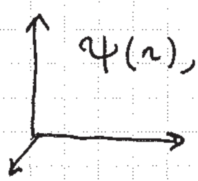
$$\Phi = \int_{\partial R} (\bar{A}_1 - \bar{A}_2)\cdot d\bar{\ell} = \int_{\partial R} (\bar{\nabla}\chi)\cdot d\bar{\ell} = 2\pi n$$

and $\Phi = 4\pi g \Rightarrow \boxed{2g = n \in \mathbb{Z}}$

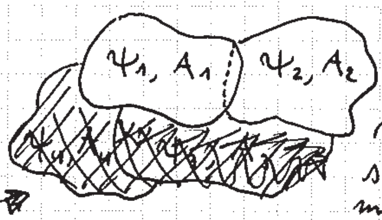
Take-home message: the monopole charge [g] determines the homotopy class of the loop $[e^{i\chi} : S^1 \to S^1]$, which describes the "gluing" of the wave functions $\Rightarrow$ a topol. invariant.

no monopole

$\Psi(r)$, $\bar{A}$ well-defined



monopole (g)



non-trivial gluing set by the winding number $2g \in \mathbb{Z}$

"fibre bundle structure" (Wu-Yang 1975)

topology: one can deform the structure but the winding / twist remains!

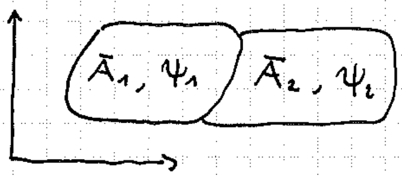
$\hookrightarrow$ monopole $\equiv$ topol. defect!

~~Link~~ with topology ②: ⚠ see 7bis/7ter.

[2nd Lecture starts here]

* First: Mention Nakahara's book "Geometry, topology and Physics"
  (homotopy groups, fibre bundles . . .)

* Previous course mentioned "fibre bundle"

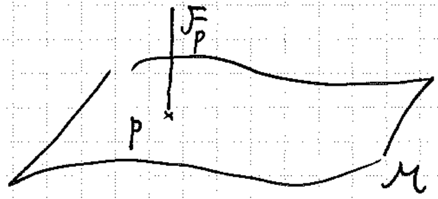


"patches" where the wave function is uniquely defined.

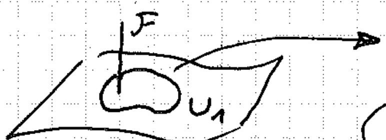
Let us be more (slightly) concrete:

Intermezzo - Fibre bundles in a nutshell

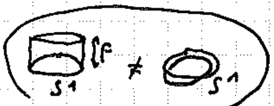
Fibre bundle $\equiv$ product of two manifolds "$\underbrace{\mathcal{M}}_{\text{base space}} \times \underbrace{\mathcal{F}}_{\text{fibre}}$" $= E$



Locally, the product is trivial: (part of the construction)



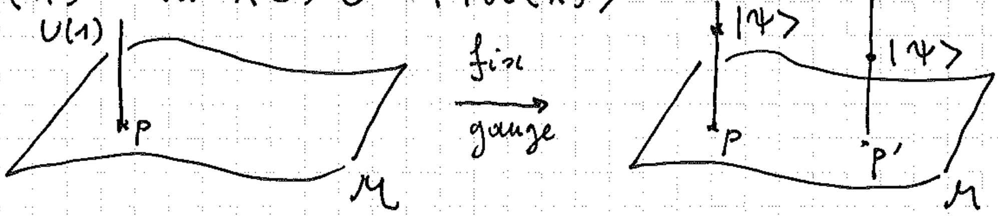
direct product $\mathcal{F} \times U_1 = \mathcal{E}|_{U_1}$



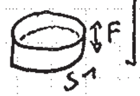
$\mapsto$ topol. classif!

Globally, $\mathcal{E} \neq \mathcal{M} \times \mathcal{F}$ (there can be a twist)

In quantum-mechanics, we often deal with $U(1)$ fibres:
$\hat{H}(\bar{\lambda})$ with $\bar{\lambda} \in \mathcal{M} \rightarrow |\Psi_{GS}(\bar{\lambda})\rangle$

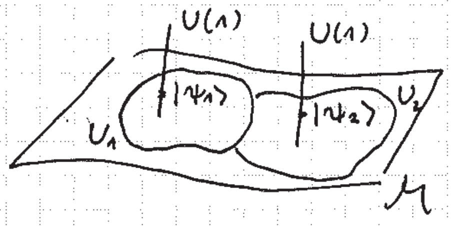


* If one can fix the gauge everywhere over $\mathcal{M}$

$\Longrightarrow \mathcal{E} = \mathcal{M} \times U(1)$: "trivial" bundle [Ex: ]

* If this is not possible (see monopole)

$\Rightarrow \mathcal{E} \neq \mathcal{M} \times U(1)$ [Ex:  Moebius



$\hookrightarrow$ this $\left\{\begin{array}{l}\text{twist}\\ \text{non-triviality}\end{array}\right.$ is measured by the $\left\{\begin{array}{l}\text{Chern number } \nu_{ch} .\\ \text{winding number } n \text{ (in this case)}\end{array}\right.$

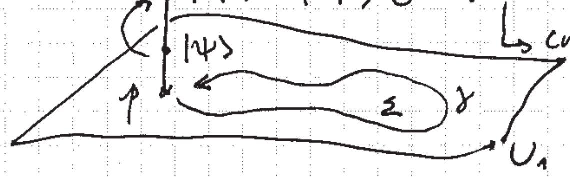
Definition of the Chern number : "Gauss Bonnet for bundles"

1°] * Introduce the curvature through parallel transport mismatch

$$|\tilde{\psi}\rangle = |\psi\rangle e^{i\oint_\gamma \bar{A}\cdot d\bar{l}} = |\psi\rangle e^{i\int_\Sigma \bar{\Omega}\cdot d\bar{S}}$$

$\hookrightarrow$ connection

$\downarrow$ $\bar{\Omega}$: curvature



$[\bar{\Omega}\cdot d\bar{S} = \Omega_{\mu\nu}\, dx^\mu \wedge dx^\nu]$

2°] * Integrate $\bar{\Omega}$ over the entire $\mathcal{M}$ (closed)

Gauss Bon. : $\dfrac{1}{2\pi}\displaystyle\int_{\mathcal{M}} \bar{\Omega}\cdot d\bar{S} =: \underbrace{\nu_{ch}}_{} \in \mathbb{Z}$

(first) Chern number (top. inv. classifying U(1) bundles).

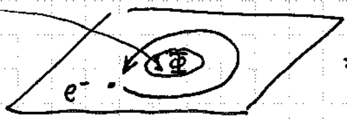
What does this mean in the monopole context ...

$\longrightarrow$ Link with topology 2.

Wu-Yang [PRD 1975] : think about the Aharonov-Bohm effect!

The Aharonov - Bohm phase :

flux



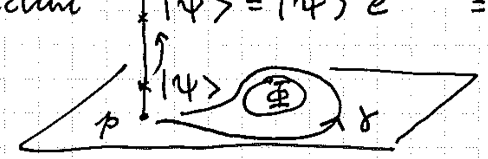
$\Rightarrow$ the wave function acquires a phase $e^{i\Phi}$

* Predicted in 1959, observed in 1960 (Chambers PRL)

Exp: "shift of an electron interference pattern due to flux"

"Fibre bundle picture" $|\tilde{\psi}\rangle = |\psi\rangle e^{i\Phi} = |\psi\rangle e^{i\oint_\gamma \bar{A}\cdot d\bar{l}} = |\psi\rangle e^{i\int_\Sigma \bar{B}\cdot d\bar{S}}$

where $\partial\Sigma =: \gamma$.

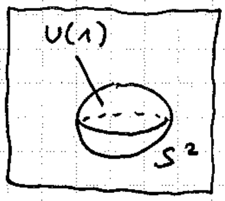


Wu & Yang [1975] : * $\bar{A}$ (gauge field) $\equiv$ connection on U(1) fibre

* $\bar{B}$ $\equiv$ curvature on this bundle

$\hookrightarrow$ Back to monopole :

We had $\underbrace{\dfrac{\Phi}{2\pi} = \dfrac{1}{2\pi}\int_{S^2} \bar{B}\cdot d\bar{S} = \overbrace{n}^{\text{winding number}}}_{} \in \mathbb{Z}$ $(=2g)$



Defines the Chern number, which classifies the fibre bundle $(S^2, U(1))$.

$n \neq 0$ ($g \neq 0 \Rightarrow$ monopole), fibre bundle $\neq S^2 \times U(1)$

(the twist being measured by n).

Cold-atom realization of a fictitious monopole :

Ref: Ray et al. Nature <u>505</u> 657 (2014).

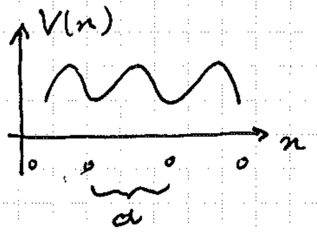
In general, the curvature is given by the field strength tensor

$\mathcal{F} = \mathcal{F}_{\mu\nu}\, dx^\mu \wedge dx^\nu$, note $[\bar{B}]^\alpha = \varepsilon^{\alpha\beta\gamma}\mathcal{F}_{\beta\gamma}$ $\quad \alpha,\beta,\gamma = x,y,z$

(we will use the tensor to describe curvature below ...)

II. Topological matter: From Bloch waves to {topological / Chern insulators
(including cold-atom measurements)

1°] Bloch theorem: introducing notations



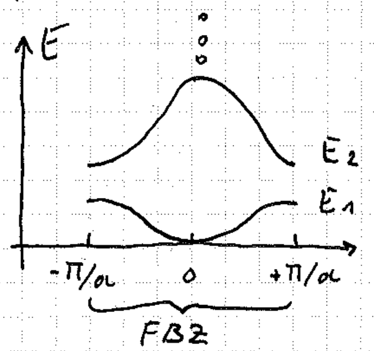
$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \; ; \quad V(\hat{x}+a) = V(\hat{x})$$

$\hookrightarrow \hat{H}|\psi\rangle = E|\psi\rangle$? $\quad \hat{H}(\hat{x}+a) = \hat{H}(\hat{x})$

$\Longrightarrow$ Bloch theorem : $\psi_{nq}(x) = e^{iqx}\, u_{nq}(x)$: modul. plane wave

$$u_{nq}(x) = u_{nq}(x+a)$$

Spectrum: bands $E_n(q)$ over q-space
$\llcorner$ band index

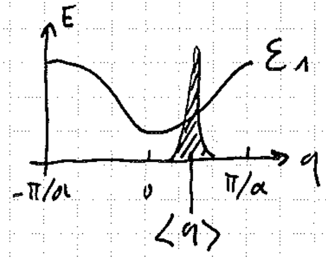


$\xrightarrow{\text{tight-binding limit}}$ $E_1(q) = -2J\cos(qa)$ — single-band description

and $\hat{H}_{eff} = -J\sum_j |j\rangle\langle j+1| + \text{h.c.}$

2°] Transport in a Bloch band: semi-classics

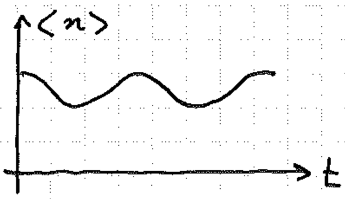
Prepare a wave-packet in a given band + act with a force F (weak)



Ashcroft-Mermin : $\begin{cases} \langle \dot{x} \rangle = \dfrac{\partial \mathcal{E}_1}{\partial q}(\langle q \rangle) & \text{: band veloc.} \\ \langle \dot{q} \rangle = F & \text{(Ehrenfest)} \end{cases}$

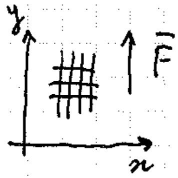
For a constant force $F \Rightarrow \langle q(t)\rangle = \langle q(t_0)\rangle + Ft$

$\Rightarrow$ Bloch oscillations



(no net current, see Joseph's lecture)

Now, let's do the same in 2D: (still in band $\mathcal{E}_1(\bar{q})$)



(1) Along the longit. direction: $\langle \dot{y} \rangle = \dfrac{\partial \mathcal{E}_1}{\partial q_y}(\langle q \rangle)$

$\hookrightarrow$ Bloch oscillation as in 1D...

(2) Along the transverse direction: $\langle \dot{x} \rangle = \underbrace{\dfrac{\partial \mathcal{E}_1}{\partial q_x}(\langle \bar{q} \rangle)}_{\text{band}} - \underbrace{F_y\, \Omega_{xy}(\langle \bar{q} \rangle)}_{\text{"anomalous velocity"}}$

where $\Omega^{(1)}_{xy}(\bar{q}) = i \left\{ \left\langle \dfrac{\partial u_1}{\partial q_x} \middle| \dfrac{\partial u_1}{\partial q_y} \right\rangle - (q_x \leftrightarrow q_y) \right\}$

Exercise: $\downarrow$ $\displaystyle = i \sum_{n' \neq 1} \frac{\langle u_1 | \partial_{q_x} \hat{H}_q | u_{n'} \rangle \langle u_{n'} | \partial_{q_y} \hat{H}_q | u_1 \rangle - (x \leftrightarrow y)}{(\mathcal{E}_{n'} - \mathcal{E}_1)^2}$

Derivation: see Appendix in Xiao - Chang - Niu RMP 2010.

Steps: (see Joseph lecture).
- Treat $\bar{F}$ as a time-dep^t gauge potential $\bar{A}(t) \Rightarrow$ transl. inv.
- $\hat{H}(\bar{q}) \longrightarrow \hat{H}[\bar{k}(\bar{q}, t)]$ with $\bar{k}(\bar{q}, t) = \bar{q} + \bar{A}(t)$
- Solve $i\partial_t |\psi\rangle = \hat{H}(t)|\psi\rangle$ with $\hat{H}[\underline{k}(t)]$ $\hookrightarrow$ slow (adiab. limit).
  for a particle initially at $|u_{1\bar{q}}\rangle$
- calculate the averaged velocity $v_\mu = \langle \psi(t) | \dfrac{\partial \hat{H}}{\partial q_\mu} | \psi(t) \rangle$

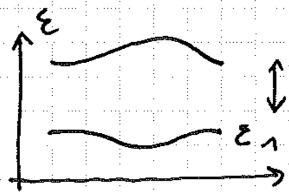
$R_q^{(1)}$: Anomalous velocity first discovered by Karplus & Luttinger (1951)

Context: "Anomalous Hall effect in ferromagnetics"

$\hookrightarrow$ "Hall current can be observed <u>without</u> a magnetic field"

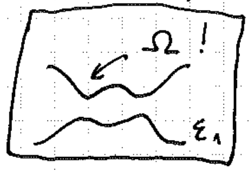
$R_q^{(2)}$: $\Omega^{(1)}_{xy} \sim \displaystyle\sum_{n' \neq 1} \frac{\langle \ldots\ldots \rangle}{(\mathcal{E}_{n'} - \mathcal{E}_1)^2}$ $\Rightarrow$ only neighboring bands contribute!

$\Rightarrow$



$\longrightarrow$ single-band TB limit: $\Omega^{(1)}_{xy} \approx 0$.

Good situation: 2-band TB models (graphene)



$R_q^{(3)}$: Result relies on adiab. motion in $\mathcal{E}^1 \Rightarrow$ requires a gap (+ weak force)

Modern interpretation : $\Omega^{(1)}_{xy}(\bar{q})$ is the Berry curvature
$\longrightarrow$ a geometric property of the band $\mathcal{E}_1$. (1980's) Simon + Berry

Anticipation of the result : [$\Omega_{xy}(\bar{q})$ is a "magnetic field" in $\bar{q}$ - space.]
A-B: [$\bar{B} \to$ curvature in real space]

$\to$ Q : what does a magnetic field do to semi-classical EoM?

* For a real magnetic field $\bar{B}$ in real space

$$\hookrightarrow \langle \dot{\bar{q}} \rangle = -(e) \langle \dot{\bar{x}} \rangle \times \bar{B} \quad : \text{Lorentz force} \quad (*)$$

* For a "magnetic field" $\bar{\Omega}$ in $\bar{q}$ - space

$$\hookrightarrow \text{dual to } (*) : \langle \dot{\bar{x}} \rangle = - \langle \dot{\bar{q}} \rangle \times \bar{\Omega} \quad (**)$$

$$\underbrace{\Omega^{\alpha}}_{\text{vector}} = \varepsilon^{\alpha\beta\gamma} \underbrace{\Omega_{\beta\gamma}}_{\text{tensor (field strength)}} \quad \text{with } \alpha, \beta, \gamma = x, y, z \ (3D)$$

2D!

$$\hookrightarrow \Omega_{xy} = (\bar{\Omega})^{z} \ ; \quad \langle \dot{\bar{q}} \rangle = \bar{F} = F_y \bar{1}_y \ ; \ \Omega_{xy} \bar{1}_z$$

$$(\bar{1}_y \times \bar{1}_z = \bar{1}_x) \Rightarrow (**) \quad \langle \dot{x} \rangle = - F_y \Omega_{xy} \quad : \text{anomalous velocity!}$$

[« The anomalous velocity is just the manifestation of the Lorentz force in $\bar{q}$ - space, due to $\bar{\Omega}$ "!]

We have to demonstrate the statement above!

$\leadsto$ introduce the Berry phase ...

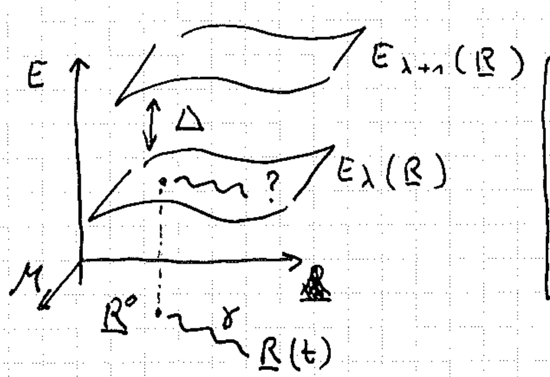
3°]. The geometric phase & the Berry curvature

General quantum system with Hamiltonian $\hat{H} = \hat{H}(\bar{R})$

$\bar{R} = (x_1, x_2, \dots) \in \mathcal{M}$: some parameters taking values on $\mathcal{M}$.

[Ex: $\bar{R} = (q_x, q_y) \in \mathrm{FBZ}$ and $\hat{H}(\bar{q})$: "Bloch Hamilt."]

* $\forall \underline{R} \in \mathcal{M}$: $\hat{H}(\underline{R})\,|\phi_\lambda(\bar{R})\rangle = E_\lambda(\underline{R})\,\underbrace{|\phi_\lambda(\underline{R})\rangle}_{\text{single-valued (fix gauge)}}$



Problem:
* Prepare state $|\psi(t_0)\rangle = |\phi_\lambda(\underline{R}^0)\rangle$
* Vary $\underline{R}(t)$ "slowly" (γ)
* How does $|\psi(t)\rangle$ evolve?

$\Rightarrow$ Solve Schrödinger eq: $i\partial_t |\psi(t)\rangle = \hat{H}[\underline{R}(t)]\,|\psi(t)\rangle$ (*)

Hypothesis: Adiabatic regime $\dot{\underline{R}} \ll$ (variation time scale $\gg 1/\Delta$) $\Rightarrow$ $\boxed{|\psi(t)\rangle \text{ stays in the } \lambda\text{-manifold } \forall t}$

We write an ansatz: $|\psi(t)\rangle = \underbrace{e^{i\theta(t)}}_{?}\,|\phi_\lambda[\underline{R}(t)]\rangle$ (**)

(**) $\to$ (*):

$$|\psi(t)\rangle = \underbrace{e^{-i\int_0^t E_\lambda(\tau)\,d\tau}}_{\text{dynamical}}\ \underbrace{e^{i\int_\gamma \underline{A}^\lambda\cdot d\underline{l}}}_{\text{geometric: only depends on path}}\ |\phi_\lambda[R(t)]\rangle$$

where $\underline{A}^\lambda = i\langle \phi_\lambda | \nabla_{\underline{R}}\, \phi_\lambda\rangle$: "Berry connection" (in band λ)

∘ This was known, but found irrelevant (gauge removed)

↳ let's analyse this!

DISCUSS LATER ⚠

[Note: dyn. phase typically vanishes in interferometric measurem^{t}:

$\Delta\phi = \int_{\gamma_1} - \int_{\gamma_2} = \oint_{loop} \bar{A}\cdot d\bar{l}$]

Let's make a gauge transformation:

$$|\phi_\lambda(\underline{R})\rangle \longrightarrow |\tilde{\phi}_\lambda(\underline{R})\rangle = e^{i\chi(\underline{R})}|\phi_\lambda(\underline{R})\rangle$$

then $\underline{A}^\lambda(\underline{R}) \longrightarrow \underline{\tilde{A}}^\lambda(\underline{R}) = \underline{A}^\lambda(\underline{R}) - \bar{\nabla}_{\underline{R}}\chi(\underline{R})$

$\Rightarrow$ « Berry connection acts like a gauge potential in $\mathcal{M}$ »

$\rightarrow$ indeed, the geometric phase can be removed!

Unless ...

* Consider a <u>closed loop</u> in $\underline{R}$-space: $\underline{R}(t_F) = \underline{R}(t_0) = \underline{R}^0$

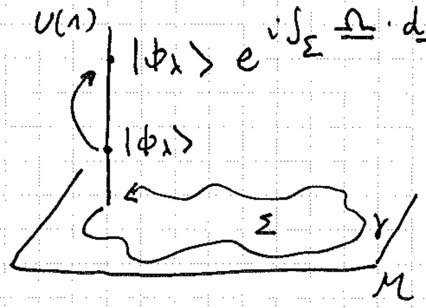
$$\Rightarrow |\psi(t_F)\rangle = e^{i\theta_{dyn}} \exp\left\{\oint_\gamma \underline{A}^\lambda \cdot d\underline{l}\right\} |\phi_\lambda(\underline{R}^0)\rangle$$

Stokes theorem: $\oint_\gamma \underline{A}^\lambda \cdot d\underline{l} = \int_\Sigma (\underbrace{\bar{\nabla}_{\underline{R}} \times \underline{A}^\lambda}_{\underline{\Omega}^\lambda}) \cdot d\bar{S}$

$\underline{\Omega}^\lambda = \bar{\nabla}_{\underline{R}} \times \underline{A}^\lambda$: Berry curvature (in band λ).

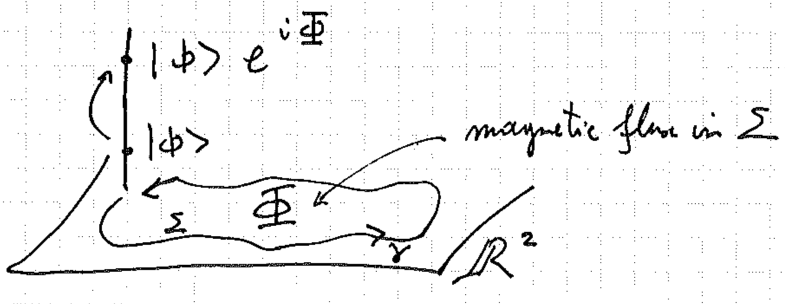
Note: $\underline{\Omega}^\lambda$ is gauge invariant! $\Rightarrow$ "magnetic field in $\mathcal{M}$".

[Berry 1984: geom. phase cannot be gauged away for cyclic evolution]



$\rightarrow$ Berry phase = "flux of $\bar{\Omega}$ penetrating the enclosed surface".

generalizes the Aharonov-Bohm effect

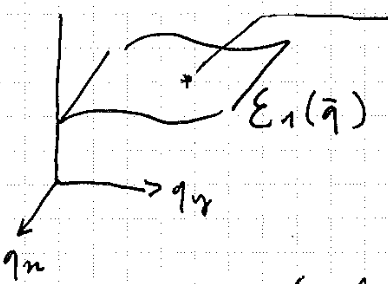


We have $\underline{A}^\lambda = i \langle \phi_\lambda | \bar{\nabla}_R \phi_\lambda \rangle$; $\underline{\Omega}^\lambda = \bar{\nabla}_R \times \underline{A}^\lambda$

$$\Rightarrow \left(\underline{\Omega}^\lambda\right)_\mu = \varepsilon^{\alpha\beta\mu}\, \Omega^\lambda_{\alpha\beta} \; ; \; \Omega^\lambda_{\alpha\beta} = i \left\{ \left\langle \frac{\partial \phi_\lambda}{\partial R_\alpha} \middle| \frac{\partial \phi_\lambda}{\partial R_\beta} \right\rangle - (\alpha \leftrightarrow \beta) \right\}.$$

* Back to Bloch bands : $\underline{R} = (q_x, q_y)$

Bloch states $|u(\bar{q})\rangle$ associated with $\hat{H}_q$.



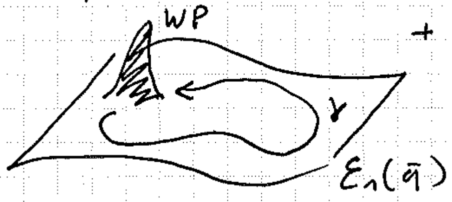
$\Rightarrow$ Berry curvature in band $\mathcal{E}_1$, at $\bar{q}$

$$\Omega^{(1)}_{xy}(\bar{q}) = i \left\{ \left\langle \frac{\partial u_1}{\partial q_x} \middle| \frac{\partial u_1}{\partial q_y} \right\rangle - (q_x \leftrightarrow q_y) \right\}$$

(which is indeed the quantity entering the anomalous velocity)

Interpretation : Consider a wave packet in band $\mathcal{E}_1(\bar{q})$

+ make a loop in $\bar{q}$-space



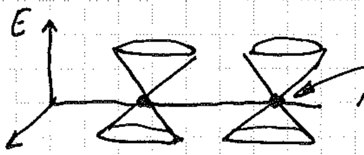
$\Rightarrow$ the WP wave function will acquire a Berry phase $\exp[i \int \underline{\Omega} \cdot d\underline{s}]$
$= \exp[i\, \Phi_{\text{Berry}}]$

Done $\Rightarrow$ Aharonov - Bohm effect in $\bar{q}$-space due to $\Omega^{(1)}_{xy}$!

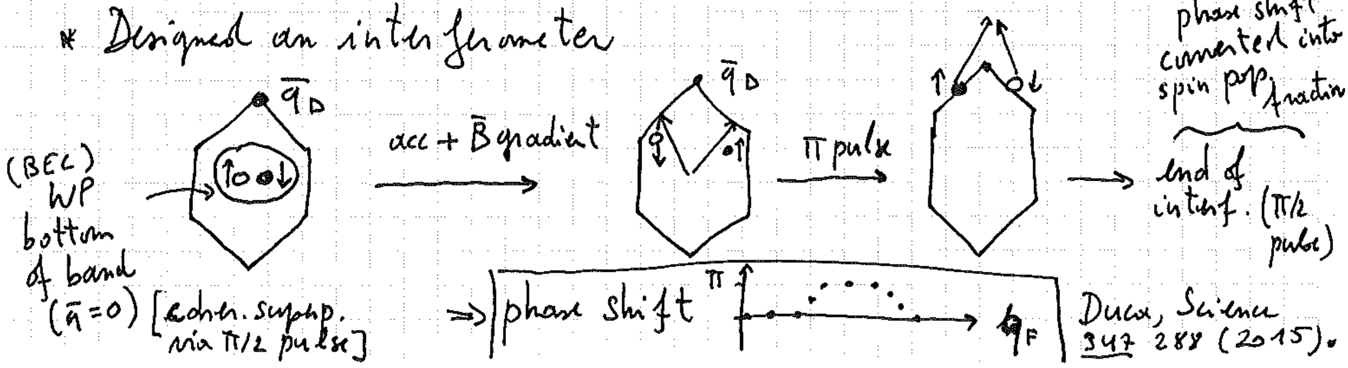
Cold-atom experiment performed in I. Bloch's lab :

* Honeycomb lattice ("graphene") for ultracold ^{87}Rb atoms

* Dispersion is Dirac-like :



* Designed an interferometer

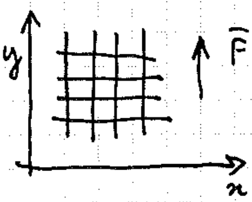


Duca, Science 347 288 (2015).

Lecture 3 starts here (after explaining pages 12 → 14)

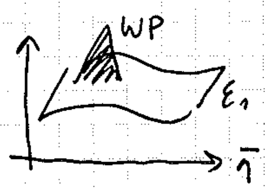
4°] Insulators and classification

We study transport in a Bloch band $\mathcal{E}_1(\bar{q})$ of a 2D lattice.



$$\text{EoM}: \begin{cases} \langle \dot{x} \rangle = \dfrac{\partial \mathcal{E}_1}{\partial q_x}(\langle \bar{q} \rangle) - F_y \, \Omega_{xy}^{(1)}(\langle \bar{q} \rangle) \\[2mm] \langle \dot{y} \rangle = \dfrac{\partial \mathcal{E}_1}{\partial q_y}(\langle \bar{q} \rangle) \end{cases}$$

and $\langle \dot{q}_x \rangle = 0$; $\langle \dot{q}_y \rangle = F_y$



Adiab. motion of WP in band $\mathcal{E}_1 \Rightarrow$ Hall drift (without magnetic field, a priori ...)

Now: We completely fill the band ("insulator").

$$\Rightarrow \bar{v}_{TOT} = \sum_{\bar{q}} \langle \dot{\bar{x}} \rangle (\bar{q}) = \frac{A}{(2\pi)^2} \int_{FBZ} d^2q \, \langle \dot{\bar{x}} \rangle (\bar{q})$$

- Observation: band velocities vanish $\displaystyle\int_{FBZ} d^2q \left(\frac{\partial \mathcal{E}_1}{\partial q_\mu} \right) = 0$

(bands are periodic in $\bar{q}$-space)

$$\Longrightarrow \boxed{v_{TOT}^{y} = 0} \quad : \text{insulator!} \left[\sigma_{yy} \propto \frac{v_{TOT}^{y}}{F_y} = 0 \right]$$

- Total transverse velocity: $\left[v_{TOT}^{x} / A = -\dfrac{F_y}{(2\pi)^2} \displaystyle\int_{FBZ} \Omega_{xy}^{(1)}(\bar{q}) \, d^2q \right]$

* Consider a 2D e^- gas:

- current density $j^x = -e \, v_{TOT}^{x} / A$
- force → electric field: $F_y = -e \, E_y$ $\qquad (h = 2\pi\hbar)$
- Chern number $\nu_{ch} = \dfrac{1}{2\pi} \displaystyle\int_{FBZ} \Omega_{xy}^{(1)}(\bar{q}) \, d^2q \in \mathbb{Z}$

$$\Rightarrow j^x = \underbrace{\left[-\frac{e^2}{h} \nu_{ch} \right]}_{\sigma_{xy} \equiv -\sigma_H \,:\, \text{Hall conductance} \,\to\, \text{quantized!}} E_y$$

$$\Rightarrow \boxed{\sigma_H = \frac{e^2}{h}\nu_{ch}} \quad \text{TKNN (1982)}$$

$\nu_{ch} \in \mathbb{Z}$: Chern number associated with the fibre bundle $(\overset{\text{FBZ}}{\mathbb{T}^2}, U(1))$.

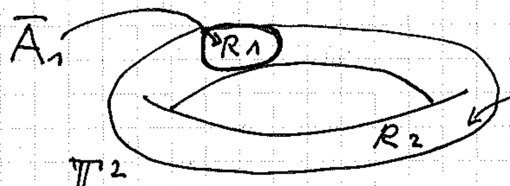
∘ Rq : $\sigma_H \neq 0 \Rightarrow \underbrace{\int_{FBZ} \Omega^{(1)}_{xy}(\bar{q})\, d^2q}_{\text{total flux of Berry curvature over } \mathbb{T}^2} \neq 0$

$\Rightarrow$ [$\sigma_H \neq 0$ implies the existence of a "monopole" in $\bar{q}$-space!]

Indeed, we have $\bar{\Omega}^{(1)} = \bar{\nabla}_{\bar{q}} \times \underbrace{\bar{A}^{(1)}}_{\text{Berry connection!}}$

$\Rightarrow \oint_{FBZ} \bar{\Omega}^{(1)} \cdot d\bar{S} \neq 0 \Leftrightarrow \bar{A}^{(1)}$ not globally defined!

$\Rightarrow$ the $U(1)$ fibre bundle is not trivial :



connection $\bar{A}_2$ $\quad \bar{A}_2 = \bar{A}_1 + \bar{\nabla}\chi$

ν_{ch} : homotopy class of the loop $e^{i\chi(\ell)} : \partial R \longrightarrow U(1)$

[see section monopole!]

Summary :

Insulator (filled band)

$\begin{cases} \sigma_{yy} = 0 \\ \sigma_H = 0 \ (\nu_{ch} = 0) \end{cases} \Rightarrow$ Trivial band insulator (Berry connect. glob. defined)

$\begin{cases} \sigma_{yy} = 0 \\ \sigma_H = (e^2/h)\nu_{ch} \neq 0 \end{cases} \Rightarrow$ Chern insulator (monopole in $\bar{q}$-space).

Note : these insulators are classified by topology (not by local order parameters)

Question: When do we observe $\sigma_H = (e^2/h)\,\nu_{ch} \neq 0$

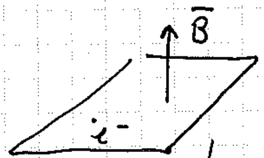
MOVE

Formally, the time-reversal of $\Omega_{xy}(\bar{q})$ is $T\Omega_{xy}(\bar{q}) = -\Omega_{xy}(-\bar{q})$

↳ [if a system has TRS $\Rightarrow \Omega_{xy}(-\bar{q}) = -\Omega_{xy}(+\bar{q})$.]

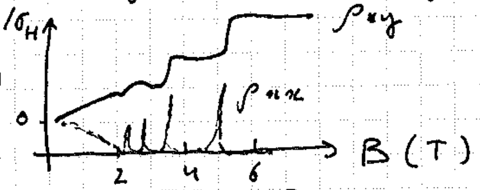
$\Rightarrow$ [$\nu_{ch} \neq 0 \iff$ TRS broken]

Historically: integer quantum Hall effect [IQH]



2D electron gas in strong (uniform) magnetic field.

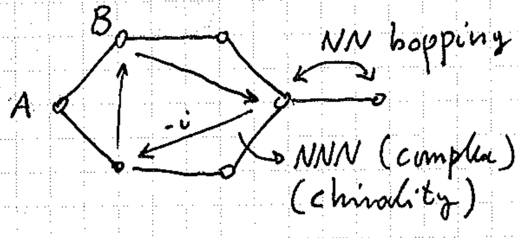
↳ von Klitzing (1980)



[Link with topology: Thouless (TKNN): 1982]
Simon 1983

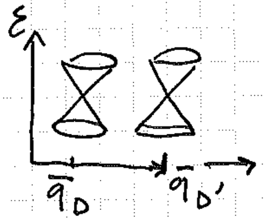
* 1988: Chern insulator WITHOUT MAGNETIC FIELD

↳ Haldane: simple 2-band (graphene) model

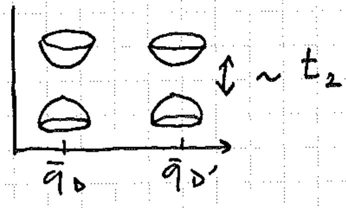


$$\hat{H} = -t_1 \sum_{NN} - t_2 \sum_{NNN} (i)^{\circlearrowleft} \qquad [+i \circlearrowleft ; -i \circlearrowright].$$

$t_2 = 0$



$t_2 \neq 0$



In momentum representation:

$$\hat{H}(\bar{q}) = \begin{bmatrix} g(\bar{q}) & f(\bar{q}) \\ f^*(\bar{q}) & -g(\bar{q}) \end{bmatrix}$$

f: NN (graphene)

g: NNN (Haldane term).

Write $\cos\theta = g/|\varepsilon(\bar q)|$; $f(\bar q) = |f|\, e^{i\varphi}$

$$\Rightarrow \hat H(q) = \underbrace{|\varepsilon(\bar q)|}_{\text{dispersion}} \begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}$$

$\varepsilon_\pm = \pm|\varepsilon(\bar q)|$

Eigenstate (lowest band): $|u_-\rangle = \begin{pmatrix} -e^{-i\varphi}\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}$.

Berry connection $\bar A^{(-)}(\bar q) = i\langle u_- | \bar\nabla u_-\rangle$

$$= \frac{1}{2}(1-\cos\theta)\,\bar\nabla_q \varphi \quad \text{(monopole } g = 1/2)$$

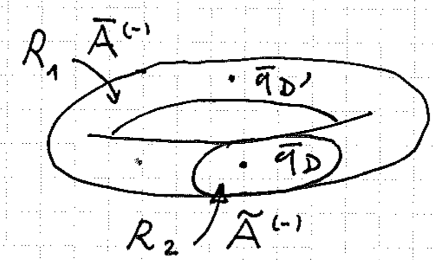
[Follow steps of monopole story]

∘ When is $\bar A^{(-)}$ singular: a). φ ill-defined $\Rightarrow f(\bar q) = 0$ (vortex).
b). $\theta = \pi \Rightarrow g(\bar q) < 0$

a) Condition $f(\bar q) = 0 \Leftrightarrow \bar q = \bar q_D$ or $\bar q_{D'}$ (vortex at Dirac p.)
b) condition $g(\bar q) < 0 \Rightarrow$ select one of them [$\bar q_D$].

Similarly $\tilde A^{(-)}(\bar q) = -\frac{1}{2}(1+\cos\theta)\,\bar\nabla_{\bar q}\varphi$ singular at $\bar q_{D'}$.

$\Rightarrow$



$$\nu_{ch} = \frac{1}{2\pi}\oint_{\partial R} (\bar A^{(-)} - \tilde A^{(-)})\cdot d\bar l$$

$$= \frac{1}{2\pi}\oint_{\partial R} \bar\nabla_{\bar q}\varphi \cdot d\bar l$$

$$= 1 \quad [f(\bar q) \text{ is a vortex of winding } 1].$$

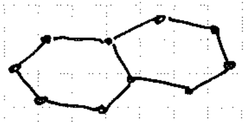
* Note: In a 2-band model with $g(\bar q)$ cst $\Rightarrow \nu_{ch} = 0$.

↳ Haldane's model: $\mathrm{sign}[g(\bar q_D)] = -\,\mathrm{sign}[g(\bar q_{D'})]$
$\Rightarrow$ exactly one singularity.
$\Rightarrow$ consequence of TRS breaking (consistent $\nu_{ch} \neq 0$).

5°] A few cold-atom experiments and beyond...

a]. Realization of the Haldane model

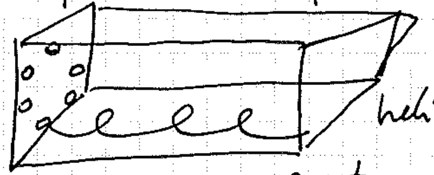
* Proposal by Aoki and Oka [PRB 2009]

graphene  ~ irradiate with circ. polar. light

$\Longrightarrow \hat{H}(t+T) = \hat{H}(t)$: Floquet engin.

chirality $\Longrightarrow \hat{H}_{eff} \equiv$ Haldane (in $\omega \to \infty$ regime).

* Implemented in photonics (Rechtsman Nature 2013)



helical wave guides $\equiv$ Aoki's proposal!

$z \equiv t$ ↳ Haldane's model for light.

* Implementation in cold atoms (Zurich / Hamburg)

Honeycomb optical lattice + circular shaking $\equiv$ Aoki's proposal. ↳ $V \sim x\cos(\omega t) + y\sin(\omega t)$ $\equiv$ circ. pol. light

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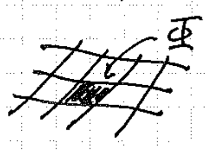
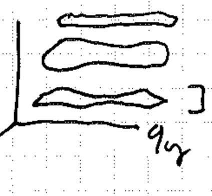
b]. Detection of the Chern number in cold atoms.

* Transport [Aidelsburger et al. Nat. Phys. 2015]

2D optical square lattice + uniform (synth.) magnetic flux



$\Longrightarrow$



] flat band with $\nu_{Ch} = 1$

Filled the band uniformly with $\boxed{\text{bosons}}$ (band mapping ✓) + applied linear gradient

Observable: transverse COM displacement $\bar{\nu}_{com} = \frac{\bar{\nu}_{TOT}}{N} = \frac{\bar{\gamma}}{n}$

$\Longrightarrow$ quantized response $\boxed{\nu_{Ch}^{exp} = 0.91(1)}$

c]. Recent work: Probing topology by "heating".

Subject a system to a circular perturbation (cir. pol. light).

$$\hat{H}(t) = \hat{H}_0 + \mathcal{E}\{x\cos(\omega t) \pm y\sin(\omega t)\}$$

What is the energy absorbed by the system?

$$P_\pm(\omega) = 4\,A_{syst}\,E^2\left[\sigma_R^{xx}(\omega) \pm \sigma_I^{xy}(\omega)\right] \quad \text{(lin. rsp.)}$$

$$\Rightarrow P_+ - P_- \propto \sigma_I^{xy}(\omega)$$

Introduce heating rates $\Gamma_\pm(\omega) = P_\pm(\omega)/\hbar\omega$

$$\Delta\Gamma = \left[\Gamma_+(\omega) - \Gamma_-(\omega)\right]/2$$

$$\Rightarrow \sigma_I^{xy}(\omega) = \hbar\omega\,\Delta\Gamma(\omega)\,/\,4\,A_{syst}\,E^2 \qquad (*)$$

Kramers-Kronig: $\sigma^{xy}(\omega\to 0) = \sigma_H = (2/\pi)\int_0^\infty \frac{\sigma_I^{xy}}{\omega}\,d\omega$ (**)

$$\Rightarrow \Delta\Gamma^{int}/A_{syst} = \frac{1}{A_{syst}}\int_0^\infty \Delta\Gamma(\omega)\,d\omega = \left(\frac{2\pi E^2}{\hbar}\right)\sigma_H$$

* Chern insulator: $\sigma_H = (e^2/h)\,\nu_{ch}$

$$\Rightarrow \left[\Delta\Gamma^{int} \text{ quantized}\right]$$ (quantized circular dichroism).

◦ Recently observed in Hamburg [Sengstock] 1805.11077

- Fermi gas in 2D honeycomb lattice
- Circular shake creates Chern bands ($\nu_{ch}=1$) + perturbation
- Measured heating rates (band mapping: pop. in upper bands)

↳ $\Delta\Gamma^{int} \sim \nu_{ch}$ reaches the quantized value in the topological regime $[\nu_{ch}^{exp} = 0.92\,(12)]$.