Energies of Bloch states

Gaps open at band edges, where light-scattering is resonant
Seeing band structure: “band mapping”

Brillouin zone structure of a 2D square lattice

Hansch/Esslinger (2001)

Esslinger ARCMP (2010)

Increasing atom number
a cube of gas
Measuring band structure

(a) Energy versus momentum plot showing the band structure with markers indicating specific points.

(b) Diagram illustrating the relationship between lattice depth, lattice amplitude modulation, adiabatic ramp, and band mapping over time.

(c) Color maps and momentum graphs showing maxima and minima.
Measuring band structure

J. Heinze, Ph.D. thesis (Hamburg 2014)
Measuring band structure

$E_R = 4.7 \, \text{kHz}$

J. Heinze, Ph.D. thesis (Hamburg 2014)

[PS — see Gabriel Chatelain’s poster on Direct cooling with amplitude modulation!]
Bloch states

\[ |\psi|^{2} \]

well defined \( \mathbf{x} \)

Discrete Fourier relations between creation operators:

\[
\hat{c}_{\ell}^{\dagger} = \frac{1}{\sqrt{M}} \sum_{n} e^{-i\mathbf{q}_{n} \cdot \mathbf{x}_{\ell}/a_{L}} \hat{c}_{q_{n}}^{\dagger}
\]

\[
\hat{c}_{q_{n}}^{\dagger} = \frac{1}{\sqrt{M}} \sum_{\ell} e^{i\mathbf{q}_{n} \cdot \mathbf{x}_{\ell}/a_{L}} \hat{c}_{\ell}^{\dagger}
\]

Wannier states

\[ |\psi|^{2} \]

well defined \( \mathbf{q} \)

sum over all \( \mathbf{q} \)
Curvature reduced at \( q=0 \), showing a larger effective mass \( m^* \).
Deep lattice limit
Isolated wells, zero tunnelling

\[ \hbar \omega_{\text{osc}} = 2\sqrt{s E_R} \]
Wannier states approach harmonic oscillator states for a deep lattice:

\[ w_0(x) \sim \exp \left[ -\frac{x^2}{2\sigma_{\text{HO}}^2} \right] \]

\[ \sigma_{\text{HO}} = s^{-1/4} a_L / \pi \]

\[ \hbar \omega_{\text{osc}} = 2\sqrt{s} E_R \]
Summary, Topic 2

- *Matter waves in a crystal of light*: single-particle eigenstates are delocalised *Bloch states*

- Band gaps show strength of *Bragg scattering* from lattice beams

- **Quasi-momentum** is not atomic momentum: atoms in a lattice are coherently dressed with photons

- The Fourier components of a band are the tunnelling coefficients

- Observations: Band mapping; Fermi surface in the band gap (a cube of gas); inter-band excitation
Fermions in Optical Lattices

- Introduction
- Fermions, statistics, & exchange
- Matter waves in crystals of light
  - The Hubbard model
- Transport
Isolation of a single band

for s=8:
\[ t=0.03E_R=6.6\text{nK}=140\text{Hz} \]
\[ \text{bandgap}=3.6E_R=0.8\mu\text{K}=16\text{kHz} \]
The Hubbard Model
The Hubbard Model

\[ \hat{H}_{\text{HM}} = -t \sum_{\langle j, \ell \rangle, \sigma} \hat{c}^\dagger_{j\sigma} \hat{c}_{\ell\sigma} + U \sum_{\ell} \hat{n}_{\ell\uparrow} \hat{n}_{\ell\downarrow} \]

- **lattice depth**
  - \( s = V/E_R \)

- **hopping**
  - \( t \)

- **on-site interaction**
  - \( U \)

- **distance between sites**
  - \( a_L \)

- **HUBBARD LN**
\[ \hat{H}_{\text{HM}} = -t \sum_{\langle j, \ell \rangle, \sigma} \hat{c}^\dagger_{j\sigma} \hat{c}_{\ell\sigma} + U \sum_{\ell} \hat{n}_{\ell\uparrow} \hat{n}_{\ell\downarrow} \]

Local interactions

\[ \hat{H}_{\text{HM}} = \sum_{q, \sigma} \mathcal{E}(q) \hat{n}_{q\sigma} + \]

Bloch waves
Bloch states

\[ |\psi|^2 \]

well defined \( q \)

Discrete Fourier relations between creation operators:

\[
\hat{C}^\dagger_{q_n} = \frac{1}{\sqrt{M}} \sum_{\ell} e^{i q_n \cdot x_\ell / a_L} \hat{C}^\dagger_{q_n}
\]

\[
\hat{C}^\dagger_{\ell} = \frac{1}{\sqrt{M}} \sum_{n} e^{-i q_n \cdot x_\ell / a_L} \hat{C}^\dagger_{q_n}
\]
Interaction term in Bloch basis

\[ \hat{H}_U = U \sum_{\ell} \hat{n}_{\ell\uparrow}\hat{n}_{\ell\downarrow} \quad \text{on-site interaction} \]

\[ \hat{H}_U = \frac{U}{M} \sum_{q_1, q_2, q_3} \hat{c}^\dagger_{q_4\uparrow} \hat{c}^\dagger_{q_3\downarrow} \hat{c}_{q_2\downarrow} \hat{c}_{q_1\uparrow} \]

where \( q_4 = q_1 + q_2 - q_3 \mod 2\pi \)

Scattering preserves quasi-momentum, not true momentum, in a lattice!
Hubbard model from two perspectives

\[ \hat{H}_{\text{HM}} = \hat{H}_0 + \hat{H}_U \]

**Localized basis:**

\[ \hat{H}_0 = \text{hopping to new site} \quad \hat{H}_U = \text{local interactions} \]

\[ \hat{H}_0 = -t \sum_{\langle j, \ell \rangle, \sigma} \hat{c}_{j\sigma}^\dagger \hat{c}_{\ell\sigma} \]

\[ \hat{H}_U = U \sum_{\ell} \hat{n}_{\ell\uparrow} \hat{n}_{\ell\downarrow} \]

**Delocalized basis:**

\[ \hat{H}_0 = \text{Bloch waves} \quad \hat{H}_U = \text{atom-atom scattering} \]

\[ \hat{H}_0 = \sum_{q, \sigma} \mathcal{E}(q) \hat{n}_{q\sigma} \]

\[ \hat{H}_U = \frac{U}{M} \sum_{q_4, q_3, q_2, q_1} \hat{c}_{q_4\uparrow}^\dagger \hat{c}_{q_3\downarrow}^\dagger \hat{c}_{q_2\downarrow} \hat{c}_{q_1\uparrow} \]
Hubbard model: observed phases, & quantum gas microscopes
Hubbard model: observed phases, & quantum gas microscopes

Bosonic SF-to-Mott Insulator transition

Quantum Gas microscopy

Phases of Fermions in optical lattices
Superfluid -to- Mott Insulator transition

\[ |\Psi_{\text{MI}}\rangle_{j=0} \propto \prod_{i=1}^{M} (\hat{a}_{i}^\dagger)^{n} |0\rangle \]

\[ |\Psi_{\text{SF}}\rangle_{U=0} \propto \left( \sum_{i=1}^{M} \hat{a}_{i}^\dagger \right)^{N} |0\rangle \]

Bosonic Mott Insulator experiments:
Munich, NIST/JQI, ETHZ, MIT, Austin, …
Bosonic quantum gas microscopes:
Harvard, MPQ, Tokyo; also local probes at Chicago, Bonn, …
Deep lattice limit

Isolated wells, zero tunnelling

\[ \hbar \omega_{\text{osc}} = 2 \sqrt{s \varepsilon_R} \]
LENSES

QE ≈ 107%

\( h_{\text{osc}} \)

occupied

empty

occupied
Sideband cooling

\[ |e\rangle \]

"resolved sideband limit": \( \Gamma \ll \omega_{osc} \)

\checkmark ion traps: \( \omega_{osc} \sim \text{MHz} \)

optical lattices:

\( \omega_{osc} = 2\sqrt{5}E_R \times E_R \sim 5 \text{ KHz} \)

\( \Gamma \sim 6 \text{ MHz} \)
Sideband cooling: EIT scheme

bare states

dressed with strong beam
Time-modulation of vertical cooling light

Light from a single atom: 20 aW
(10^{11} times weaker than Cz)
EIT + Raman!
Combined EIT+Raman sideband cooling
Observation of commensurate order

cf. time-of-flight: (no signature)

Fermionic Mott Insulator experiments:
ETHZ, Munich, Hamburg, MIT, UIUC, Harvard, Bonn, Princeton …
Bose Hubbard model

\[ \hat{H}_{HM} = \hat{H}_0 + \hat{H}_U \]

\[ \hat{H}_0 = \text{hopping to new site} \quad \hat{H}_U = \text{local interactions} \]

\[ -t \sum_{\langle j, \ell \rangle} \hat{b}_j^\dagger \hat{b}_\ell \]

\[ U \sum_{\ell} \hat{n}_\ell (\hat{n}_\ell - 1) \]

\[ \hat{H}_0 = \text{Bloch waves} \quad \hat{H}_U = \text{atom-atom scattering} \]

\[ \sum_q E(q) \hat{n}_q \]

\[ \frac{U}{M} \sum_{q_1, q_2, q_3} \hat{b}_{q_4}^\dagger \hat{b}_{q_3}^\dagger \hat{b}_{q_2} \hat{b}_{q_1} \]
Munich (2010)

No local signature of BEC.

Munich (2002)

time-of-flight signature:

phase coherence!
Hubbard model: observed phases, & quantum gas microscopes

Bosonic SF-to-Mott Insulator transition

Quantum Gas microscopy

Phases of Fermions in optical lattices
Phase diagram of the Fermi-Hubbard model at half filling

- **BEC regime**
- **BCS regime**
- **Slater regime**
- **Mott regime**

- Normal fluid (NF)
- Superfluid (SF)
- Correlated Fermi liquid (CFL)
- Mott insulator (MI)
- Paramagnetic (PM)

Formulas:
- $k_B T_N \approx 4t^2/|U|$
- $k_B T_p/M \approx 6te^{-7t/|U|}$
- $k_B T_\text{SF} \approx 4t^2/|U|$
- $k_B T \approx |U|$
Phase diagram of the Fermi-Hubbard model at half filling

- Normal fluid (NF)
- Superfluid (SF)
- Correlated Fermi liquid (CFL)
- Mott insulator (MI)
- Paramagnetic (PM)

Temperature vs. $k_B T / t$ and $U / t$ for attractive and repulsive interactions.

- BEC regime
- BCS regime
- Slater regime
- Mott regime

$k_B T_p \approx 6t e^{-7t/U}$

$U / t$

$k_B T_M \approx U$

$U / t$

$k_B T_p / M \approx 4t^2 / |U|$

$k_B T_{SF} \approx 4t^2 / |U|$

$k_B T_N \approx 4t^2 / |U|$

Néel temperature

Tarruell & Sanchez-Palencia (2018)
Fermionic Mott Insulator experiments:
ETHZ, Munich, Hamburg, MIT, UIUC, Harvard, Bonn, Princeton …
Insulating states of fermions

A Metal: \( J > 0 \)

- Delocalized atoms

B Mott-Insulator: \( U \ll E_t \ll 12J \)

- Localized atoms

C Band-Insulator: \( E_t \gg 12J, U \)

- Distance from trap center \( r(d) \)

\( n_{0,\sigma} = 0.5, p = 0 \)

\( n_{0,\sigma} = 1, p \rightarrow 1 \)
Evidence for insulating state of fermions: (1) reduced compressibility
Evidence for insulating state of fermions: (2) fewer doublons

\[ U/(6J) = 0, V_0 = 7E_r \]

\[ U/(6J) = 4.8, V_0 = 7E_r \]

\[ U/(6J) = 19, V_0 = 12E_r \]

\[ U/(6J) = 25, V_0 = 12E_r \]
Evidence for insulating state of fermions: (3) Mott gap
Direct observation of insulating order parameter

similar results:
Munich, Harvard, MIT, Princeton...
Trap effect: change local chemical potential

A

Band Insulator

Metal

Mott Insulator

B

\[ \frac{\tau_{\text{det}}}{\tau_{\text{det}}} = 2.5(1) \]

\[ \frac{\tau_{\text{det}}}{\tau_{\text{det}}} = 3.8(2) \]

\[ \frac{\tau_{\text{det}}}{\tau_{\text{det}}} = 15.3(7) \]
Phase diagram of the Fermi-Hubbard model at half filling

- Normal fluid (NF)
- Superfluid (SF)
- Correlated Fermi liquid (CFL)
- Mott insulator (MI)
- BCS regime
- BEC regime
- Slater regime
- Mott regime
- Néel temperature $k_B T_N \approx 4t^2/|U|$
- Pairing transition temperature $k_B T_{p/M} \approx 6t e^{-7t/U}$

Tarruell & Sanchez-Palencia (2018)
colder: AFM!
PS — see Joannis Koepsell’s poster on imaging polarons in 2D with this technique!
Summary, Topic 3

• The Hubbard Model (HM) is the minimal model for interacting fermions in an optical lattice

• HM in localized basis = hopping + on-site interaction
  HM in quasi-momentum basis = KE + scattering

• Rapid progress in quantum simulation of the Fermi Hubbard model, driven by new quantum in-situ detection techniques

• Observations of Mott insulators, band insulators, anti-ferromagnetism, and more.

• Frontiers: even lower temperature; away from half-filling; and spin imbalance.
Fermions in Optical Lattices

- Introduction
- Fermions, statistics, & exchange
- Matter waves in crystals of light
- The Hubbard model
- Transport