FERMI GASES IN TWO DIMENSIONS

Les Houches predoc school

9–12 October 2018

[4 × 1.5 hour lectures]

FERMI GASES IN TWO DIMENSIONS

#L.1.

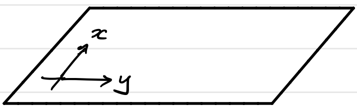
- Why study fermions in 2D?

  - 2D systems are technologically useful
    e.g., 2D structures form the basis of electronics (transistors)
    Graphene (Nobel Prize 2010)
    Layered materials such as high-temperature superconductors (Nobel Prize 1987)

  - 2D systems are challenging to treat theoretically
    ↳ Standard approaches (mean-field theory) not reliable, no access to exact solutions like in 1D.
    ∴ we require benchmarks for theory → cold atoms

  - Low dimensions can lead to interesting physics!

- The "pure" 2D system
  ↳ Consider a gas of particles confined to a 2D plane:



Area $= L^2$

For the moment, we forget about how this is achieved in practice, and we just focus on the idealized 2D system. But we will return to this point later.

[Thickness of plane needs to be smaller than all lengthscales of interest, e.g. interparticle spacing]

- Properties of 2D

(i) Density of states is <u>constant</u> for a <u>uniform</u> system

— Consider single type of particle with mass m

— States have definite momentum $\vec{k}$ and energy

$$\varepsilon_k = \frac{|\vec{k}|^2}{2m} \equiv \frac{k^2}{2m} \quad (\text{we set } \hbar = 1)$$

$\therefore$ D.O.S. at energy ε is: $\rho(\varepsilon) = \frac{1}{L^2}\sum_{\vec{k}} \delta(\varepsilon - \varepsilon_k)$

↳ defined so that $\frac{1}{L^2}\sum_{\vec{k}} = \int d\varepsilon\, \rho(\varepsilon)$

Now since $\Delta k = \frac{2\pi}{L}$, $\frac{1}{L^2}\sum_{\vec{k}} = \frac{1}{(2\pi)^2}\sum_{\vec{k}}(\Delta k)^2 \xrightarrow[L\to\infty]{} \frac{1}{(2\pi)^2}\int d^2k$

$$\begin{aligned}\therefore \rho(\varepsilon) &= \frac{1}{(2\pi)^2}\int d^2k\, \delta(\varepsilon - \varepsilon_k) \\ &= \frac{1}{2\pi}\int dk\, k\, \delta(\varepsilon - \varepsilon_k) \qquad \leftarrow d\varepsilon_k = \frac{k}{m}dk \\ &= \frac{m}{2\pi}\underbrace{\int d\varepsilon_k\, \delta(\varepsilon - \varepsilon_k)}_{1}\end{aligned}$$

$$\boxed{\therefore \rho(\varepsilon) = \frac{m}{2\pi}}$$

[Contrast with other dimensions:

$$\rho_{3D}(\varepsilon) = \frac{m^{3/2}}{\sqrt{2}\pi^2}\sqrt{\varepsilon} \quad ; \quad \rho_{1D}(\varepsilon) = \frac{1}{\pi}\sqrt{\frac{m}{2\varepsilon}}$$

]

— The constant D.O.S. means that there are <u>more</u> states at low energies compared to 3D.

— Consequences for bound states, phase transitions, ...

• Example: critical temperature for BEC

Boson density $n = \frac{1}{L^2} \sum_{\vec{k}} \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1}$; μ = chemical pot. $\beta = 1/T$, $(k_B = 1)$

$$= \int d\varepsilon \, \rho(\varepsilon) \frac{1}{e^{\beta(\varepsilon - \mu)} - 1}$$

i.e. $$n = \frac{m}{2\pi} \int_0^\infty d\varepsilon \left[\frac{e^{\beta(\varepsilon-\mu)}}{e^{\beta(\varepsilon-\mu)} - 1} - 1 \right]$$

$$= \frac{m}{2\pi} \left[\frac{1}{\beta} \log\left(e^{\beta(\varepsilon-\mu)} - 1\right) - \varepsilon \right]_0^\infty$$

$$= -\frac{m}{2\pi\beta} \log\left(\frac{e^{-\beta\mu}}{e^{-\beta\mu} - 1} \right)$$

$$\therefore \; n = \frac{m}{2\pi\beta} \log\left(\frac{1}{1 - e^{\beta\mu}} \right)$$ $\Rightarrow$ diverges as $\mu \to 0^-$

∴ can accommodate any number of bosons in the thermal cloud at finite temperature

∴ no BEC when $T > 0$

⌜ Alternatively, one can write:

$$k_B T_c = \lim_{z \to 1^-} \left[\frac{2\pi n}{m} \frac{1}{\log\left(\frac{1}{1-z}\right)} \right] = 0$$

where $z \equiv e^{\beta\mu}$ is the fugacity ⌟

(ii) A system with contact interactions is classically <u>scale invariant</u>

- Consider the Hamiltonian for many particles:

$$\hat{H} = \sum_i \frac{p_i^2}{2m} + g \sum_{i<j} \delta^{(2)}(\vec{r}_i - \vec{r}_j)$$

The scaling transformation $\vec{r} \rightarrow \lambda \vec{r}$ gives:

$$p^2 \longrightarrow \frac{1}{\lambda^2} p^2 \qquad (\text{since } p = -i\nabla)$$

$$\delta^{(2)}(\vec{r}) \longrightarrow \frac{1}{\lambda^2} \delta(\vec{r})$$

∴ we obtain a simple scaling of the Hamiltonian:

$$\boxed{\hat{H} \longrightarrow \frac{1}{\lambda^2} \hat{H}}$$

∴ the physics does not depend on any lengthscale (e.g., interparticle spacing)

- But the contact interaction is ill-defined in the quantum problem. Once it is renormalized, the classical symmetry is broken and a lengthscale is introduced.

↳ "Quantum Anomaly"

(iii) A two-body bound state exists for arbitrarily weak (s-wave) attraction in 2D

[Landau + Lifshitz]

→ introduces lengthscale (size of bound state)

- Consider two distinguishable particles in centre-of-mass frame.
  Relative wave function $\psi(r)$ satisfies:

$$-\frac{\nabla^2\psi}{2m_r} + v(r)\psi(r) = E\psi(r);$$

v = interaction
m_r = reduced mass
E = energy

Transform to momentum space:

$$v(r) = \frac{1}{L^2}\sum_k \tilde{v}(q)e^{iq\cdot r}; \quad \psi(r) = \frac{1}{L}\sum_k \tilde{\psi}(k)e^{ik\cdot r}$$

$$\frac{k^2}{2m_r}\tilde{\psi}(k) + \frac{1}{L^2}\sum_{k'}\tilde{v}(k-k')\tilde{\psi}(k') = E\tilde{\psi}(k)$$

Now define $\eta(k) = -\frac{1}{L^2}\sum_{k'}\tilde{v}(k-k')\tilde{\psi}(k')$

and consider bound states where $E = -\varepsilon_B < 0$.

$\therefore$ we have:

$$\tilde{\psi}(k) = \frac{1}{k^2/2m_r + \varepsilon_B}\eta(k)$$

i.e., $$\eta(k) = -\frac{1}{L^2}\sum_{k'}\frac{\tilde{v}(k-k')}{k'^2/2m_r + \varepsilon_B}\eta(k') \qquad (*)$$

Now consider regime where particles are nearly unbound:

$$\varepsilon_B \to 0, \quad \tilde{\psi}(k) \to \delta_{k,0}$$

$$\therefore \eta(k) \to -\frac{1}{L^2}\tilde{v}(k)$$

$\therefore$ (*) becomes: $$\tilde{v}(k) \simeq -\frac{1}{L^2}\sum_{k'}\frac{\tilde{v}(k-k')}{k'^2/2m_r + \varepsilon_B}\hat{v}(k')$$

Now divide sum into small k and large k:

$$\tilde{v}(k) \simeq -\frac{1}{L^2}\left[\tilde{v}(k)\tilde{v}(0)\sum_{k'<k_0}\frac{1}{k'^2/2m_r+\varepsilon_B}\right.$$

$$\left.+\sum_{k'>k_0}\frac{\tilde{v}(k-k')\tilde{v}(k')}{k'^2/2m_r+\varepsilon_B}\right]$$

where k_0 is typical range over which $\tilde{v}(k)$ varies.

$\therefore$ we have:

$$\tilde{v}(k) \simeq \frac{m_r}{2\pi}\tilde{v}(k)\tilde{v}(0)\log(\varepsilon_B) + \ldots$$

Now $\log(\varepsilon_B) \longrightarrow -\infty$ as $\varepsilon_B \to 0$, so first term dominates and we have:

$$\tilde{v}(k) \simeq \frac{m_r}{2\pi}\tilde{v}(k)\tilde{v}(0)\log(\varepsilon_B)$$

i.e. $\dfrac{2\pi}{m_r}\dfrac{1}{\tilde{v}(0)} \simeq \log(\varepsilon_B)$

$\therefore$ $\varepsilon_B \to 0$ only when $\tilde{v}(0) \longrightarrow 0$

Thus, there is always a bound state when $\tilde{v}(0) < 0$.

$\longrightarrow$ Argument applies to general interaction

- Qu: what happens in 3D?

- Renormalization of short-range (contact) interactions $\rightarrow v(r) = g\,\delta(r)$

  - Equation for bound state with binding energy ε_B:

$$\frac{k^2}{2m_r}\tilde{\psi}(k) + \frac{g}{L^2}\sum_{k'}\tilde{\psi}(k') = -\varepsilon_B\,\tilde{\psi}(k)$$

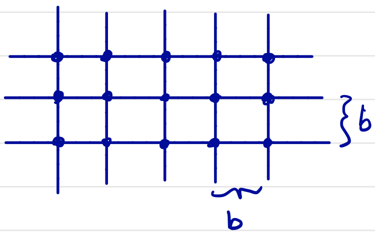
$$\hookrightarrow -\frac{1}{g} = \frac{1}{L^2}\sum_k \frac{1}{k^2/2m_r + \varepsilon_B} \quad \Big\} \text{ UV divergent}$$

By imposing a cut-off "by hand," we obtain:

$$-\frac{1}{g(\Lambda)} = \frac{1}{2\pi}\int_0^{\Lambda} dk\, \frac{k}{k^2/2m_r + \varepsilon_B} = \frac{m_r}{2\pi}\log\left(\frac{\Lambda^2/2m_r + \varepsilon_B}{\varepsilon_B}\right)$$

$\hookrightarrow$ allows us to write bare parameter $g(\Lambda)$ in terms of physical parameter, ε_B.

- Alternatively one could use the lattice model (see Werner's lectures and 1103.2851)



- In this case, momenta are restricted to the 2D Brillouin zone:

$$\vec{k} \in \left(-\frac{\pi}{b}, \frac{\pi}{b}\right)^2$$

and the interaction is now defined on one lattice site.

- Scattering with a short-range potential
  - Two-body scattering problem in c.o.M. frame:

$$\underbrace{(\hat{H}_0}_{\text{kinetic energy}} + \underbrace{\hat{V})}_{\text{2-body interaction}} |\psi\rangle = E|\psi\rangle$$

For a nice discussion of the Lippmann-Schwinger equation, see:
http://www.thp.uni-koeln.de/alexal/pdf/advqm.pdf

General solution (Lippmann-Schwinger):

unscattered wave $\hat{H}|\psi_0\rangle = E|\psi_0\rangle$

$$|\psi\rangle = \underbrace{(E-\hat{H}_0+i0)^{-1}}_{\text{Green's function } \hat{G}_0(E+i0)}\hat{V}|\psi\rangle + |\psi_0\rangle$$

Define T matrix:

$$|\psi\rangle = (1+\hat{G}_0\hat{T})|\psi_0\rangle$$

Now $|\psi\rangle = \hat{G}_0\hat{V}|\psi\rangle + |\psi_0\rangle$, i.e., $|\psi\rangle = (1-\hat{G}_0\hat{V})^{-1}|\psi_0\rangle$

$$\therefore \quad 1+\hat{G}_0\hat{T} = (1-\hat{G}_0\hat{V})^{-1} \Rightarrow G_0T - \hat{G}_0\hat{V} - \hat{G}_0\hat{V}\hat{G}_0\hat{T} = 0$$

$$\boxed{\therefore \quad \hat{T} = \hat{V} + \hat{V}\hat{G}_0\hat{T}}$$

↳ $\langle k_f|\hat{T}|k_i\rangle$ is proportional to the scattering amplitude [see Walraven's lectures]

For the case of contact interactions, we have:

$$\langle k_f|\hat{T}(E+i0)|k_i\rangle = g + g^2\frac{1}{L^2}\hat{\sum_q}\langle q|\frac{1}{E-\hat{H}_0+i0}|q\rangle + \cdots$$

$$= \left(\frac{1}{g} - \Pi(E)\right)^{-1}$$

where $\Pi(E) = \frac{1}{L^2}\hat{\sum_q}\langle q|\frac{1}{E-\hat{H}_0+i0}|q\rangle = \frac{1}{L^2}\hat{\sum_q}\frac{1}{E-q^2/2m_r+i0}$

Using the expression for $g(\Lambda)$, we finally obtain (in the limit $\Lambda \to \infty$):

$$T(E) \equiv \langle k_f | \hat{T}(E+i0) | k_i \rangle = \frac{2\pi}{m_r} \frac{1}{\log(\varepsilon_B/E) + i\pi}$$

* Notable features:
  - There is always a pole, $E = -\varepsilon_B$, i.e., a bound state
  - The scattering goes to zero when $E \to 0$ (unlike in 3D)
  - Non-monotonic function of energy with maximum in $\mathrm{Re}[T(E)]$ at $E = \varepsilon_B$

[Ref: 1408.2737]

* Scattering amplitude:

$$f(k) = 2m_r\, T(k^2/2m_r) \qquad k = \text{relative momentum}$$

$$= \frac{4\pi}{\log(2m_r \varepsilon_B/k^2) + i\pi}$$

i.e., $$\boxed{f(k) = \frac{2\pi}{\log(1/k a_{2D}) + i\pi/2}}, \quad \varepsilon_B = \frac{1}{2m_r a_{2D}^2}$$

- 2D scattering length $a_{2D} > 0$ (one convention)

- Logarithmic dependence on length scale a_{2D}

∴ weak violation of classical scale invariance

→ Quantum anomaly (see, e.g., Olshanii et al., 1006.1072)

#L.2

• <u>BCS–BEC crossover</u> in 2D [Ref.: 1402.5171]

Consider two-component $(\uparrow, \downarrow)$ Fermi gas in uniform space
- equal masses $(m_\downarrow = m_\uparrow \equiv m)$
- equal spin populations $(n_\uparrow = n_\downarrow \equiv n/2 ;\ \mu_\uparrow = \mu_\downarrow \equiv \mu)$
- attractive contact interactions (s wave)

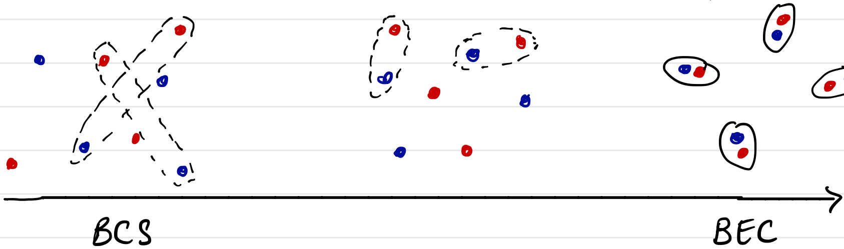
• Hamiltonian:

$$\hat{H} - \mu \hat{N} = \hat{H}_0 - \mu \hat{N} + \hat{H}_{int}$$

where $\hat{H}_0 - \mu \hat{N} = \sum_{k,\sigma} (\varepsilon_k - \mu) c^\dagger_{k\sigma} c_{k\sigma}$

$$\hat{H}_{int} = \frac{g}{\Omega} \sum_{k,k',q} c^\dagger_{k+q/2\uparrow} c^\dagger_{-k+q/2\downarrow} c_{-k'+q/2\downarrow} c_{k'+q/2\uparrow}$$

$\Omega \leftarrow$ system volume/area



In 3D, the BCS regime is when $a_s < 0$, i.e, <u>no</u> bound state [Chevy lect.]

How do we achieve a crossover in 2D, where there is <u>always</u> a bound state?

→ By varying the ratio $\varepsilon_B / \varepsilon_F$: $\varepsilon_B/\varepsilon_F \ll 1$, BCS
$\varepsilon_B/\varepsilon_F \gg 1$, BEC

→ Can have a <u>density</u>-driven crossover

(ii) Variational wave function approach

- Take boson operator $b_q^\dagger = \sum_k \varphi_k \, c^\dagger_{k+q/2\uparrow} \, c^\dagger_{-k+q/2\downarrow}$
- Coherent state:

$$|\psi\rangle = N e^{\lambda b_0^\dagger} |0\rangle = N e^{\lambda \sum_k \varphi_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}} |0\rangle$$

normalization

$$\therefore \; |\psi\rangle = \prod_k \left(u_k + v_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \right) |0\rangle$$

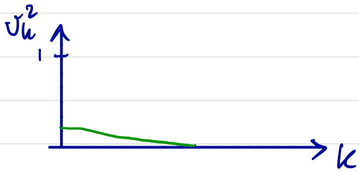
where $v_k / u_k = \lambda \varphi_k$, $N = \prod_k u_k$, $u_k^2 + v_k^2 = 1$

↳ why real?

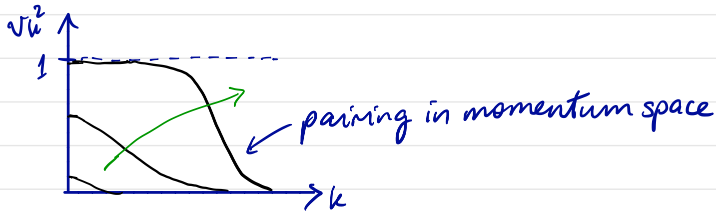
↳ BCS wave function is smoothly connected to coherent state of bosons (BEC)

Note that $\langle \psi | c^\dagger_{k\sigma} c_{k\sigma} | \psi \rangle = v_k^2 \longrightarrow$ momentum distribution for each spin

Qu: how does v_k^2 look in BEC limit?



Increasing density:



- Free energy:

$$F = \langle\psi|(\hat{H}-\mu\hat{N})|\psi\rangle = 2\sum_k (\varepsilon_k-\mu)v_k^2 + \frac{g}{\Omega}\sum_{k,k'} v_k u_k v_{k'} u_{k'}$$
$$+ \underbrace{\frac{g}{\Omega}\sum_{k,k'} v_k^2 v_{k'}^2}$$

$gn^2 \to 0$ as $g\to 0$ and $\Lambda \to \infty$

- Minimize F w.r.t. u_k, v_k at fixed μ:
  ↳ with constraint $u_k^2+v_k^2=1 \to v_k = \sin\theta_k,\ u_k = \cos\theta_k$

$$(*)\quad 2(\varepsilon_k-\mu)u_k v_k + (u_k^2-v_k^2)\frac{g}{\Omega}\sum_{k'} u_{k'}v_{k'} = 0$$

In the limit $v_k \to 0$, we recover the 2-body equation:

$$2\varepsilon_k v_k + \frac{g}{\Omega}\sum_{k'} v_{k'} \simeq 2\mu v_k, \quad \therefore\ 2\mu \simeq -\varepsilon_B$$

Defining $\Delta = -\frac{g}{\Omega}\sum_k u_k v_k$, $\xi_k = \varepsilon_k - \mu$, Eq. (*) then gives:

$$\frac{u_k v_k}{u_k^2 - v_k^2} = \frac{\Delta}{2\xi_k} = \frac{1}{2}\frac{\sin 2\theta}{\cos 2\theta}$$

$$\hookrightarrow \sin 2\theta = \frac{\Delta}{\sqrt{\xi_k^2+\Delta^2}} \to u_k v_k = \frac{\Delta}{2\sqrt{\xi_k^2+\Delta^2}}$$

$$\cos 2\theta = u_k^2 - v_k^2 = \frac{\xi_k}{\sqrt{\xi_k^2+\Delta^2}}$$

$$\therefore\ \Delta = -\frac{g}{\Omega}\sum_k \frac{\Delta}{2\sqrt{\xi_k^2+\Delta^2}}$$

Likewise we can solve for v_k^2 in terms of Δ and ξ_k
thus we end up with the equations:

(1) $$-\frac{1}{g} = \frac{1}{\Omega}\sum_k \frac{1}{2\sqrt{\xi_k^2+\Delta^2}}$$ "Gap eq'n"

(2) $$n_\sigma = \frac{1}{2\Omega}\sum_k \left(1 - \frac{\xi_k}{\sqrt{\xi_k^2+\Delta^2}}\right)$$ Density/number eq'n

* Note that Δ gives a measure of fermion pairing, since it is easy to show that: $\Delta = -\frac{g}{\Omega}\sum_k \langle c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}\rangle$

- Replace $g(\Lambda)$ using T matrix
  - For 2D, we replace with ε_B:

$$\frac{1}{g} = \underbrace{T^{-1}(-\varepsilon_B)}_{0} + \Pi(-\varepsilon_B)$$

i.e. $\frac{1}{g} = \Pi(-\varepsilon_B)$ (same as before)

  - For 3D, we instead have:

$$\frac{1}{g} = \underbrace{T^{-1}(0)}_{\frac{m}{4\pi a_s}} + \Pi(0)$$

$\frac{m}{4\pi a_s} \leftarrow$ 3D scattering length

- Solving (1) and (2) in 2D gives:

$$\Delta = \sqrt{2\varepsilon_F\varepsilon_B}$$

$$\mu = \varepsilon_F - \varepsilon_B/2$$

$\left[\text{with } \varepsilon_F = \frac{k_F^2}{2m} = \frac{2\pi n_\sigma}{m}\right]$

- In the BCS limit, $\mu \simeq \varepsilon_F$; while in the BEC limit $\mu \to -\varepsilon_B/2$
  (high density) (low density)

Qu: what are the dimer-dimer interactions in the BEC limit?

↳ Dimer chemical potential $\mu_d = 2\mu + \varepsilon_B \simeq g_{dd}\, n_d$

∴ since we have $\mu_d = \varepsilon_F$ and $n_d = n_\sigma$, we get:

$$\boxed{g_{dd} = \frac{4\pi}{m}}$$

→ classically scale invariant; theory misses quantum anomaly!

(ii) Alternative derivation – mean-field approach

- Assume fermion pairing dominates, such that we can describe it using the mean field:

$$\Delta = -\frac{g}{\Omega}\sum_k \langle c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}\rangle \qquad \text{(like previously)}$$

Write operators as:

$$-\frac{g}{\Omega}\sum_k c^\dagger_{k+q/2\uparrow}\, c^\dagger_{-k+q/2\downarrow} = \Delta\,\delta_{q,0} + \delta\hat{\Delta}_q$$

where $\delta\hat{\Delta}_q = -\Delta\,\delta_{q,0} - \frac{g}{\Omega}\sum_k c^\dagger_{k+q/2\uparrow}\, c^\dagger_{-k+q/2\downarrow}$

↳ assumed small

- Now expand $\hat{H}_{int}$ up to linear order in $\delta\hat{\Delta}$:

$$\hat{H}_{int} = \frac{g}{\Omega}\sum_{k,k',q} c^\dagger_{k+q/2\uparrow} c^\dagger_{-k+q/2\downarrow} c_{-k'+q/2\downarrow} c_{k'+q/2\uparrow}$$

$$= \frac{\Omega}{g}\sum_q \left|\Delta\delta_{q,0} + \delta\hat{\Delta}_q\right|^2$$

$$\simeq \frac{\Omega}{g}\Delta^2 + \Delta\delta_{q,0}\left(\delta\hat{\Delta}_q + \delta\hat{\Delta}^\dagger_q\right)$$

$$= -\frac{\Omega}{g}\Delta^2 - \Delta\sum_k c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow} - \Delta\sum_k c_{-k\downarrow}c_{k\uparrow}$$

$\therefore$ we have mean-field Hamiltonian:

$$\hat{H}_{MF} = -\frac{\Omega}{g}\Delta^2 + \sum_k(\varepsilon_k-\mu)c^\dagger_{k\sigma}c_{k\sigma} - \Delta\sum_k c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow} - \Delta\sum_k c_{-k\downarrow}c_{k\uparrow}$$

$$= -\frac{\Omega}{g}\Delta^2 + \sum_k \Psi^\dagger_k\begin{pmatrix}\varepsilon_k-\mu & \Delta\\ \Delta & \mu-\varepsilon_k\end{pmatrix}\Psi_k + \sum_k(\varepsilon_k-\mu)$$

where $\Psi^\dagger_k = (c^\dagger_{k\uparrow}, c_{-k\downarrow})$

Diagonalize Hamiltonian using transformation:

$$\begin{pmatrix}\gamma^\dagger_{k\uparrow}\\ \gamma_{-k\downarrow}\end{pmatrix} = \begin{pmatrix}u_k & -v_k\\ v_k & u_k\end{pmatrix}\begin{pmatrix}c^\dagger_{k\uparrow}\\ c_{-k\downarrow}\end{pmatrix} \qquad \text{(exercise!)}$$

i.e. $$\hat{H}_{MF} = -\frac{\Omega}{g}\Delta^2 + \sum_k(\varepsilon_k-\mu) + \sum_k\left(E_k\gamma^\dagger_{k\uparrow}\gamma_{k\uparrow} - E_k\gamma_{k\downarrow}\gamma^\dagger_{k\downarrow}\right)$$

$$= -\frac{\Omega}{g}\Delta^2 + \sum_k(\varepsilon_k-\mu-E_k) + \sum_k E_k\left(\gamma^\dagger_{k\uparrow}\gamma_{k\uparrow} + \gamma^\dagger_{k\downarrow}\gamma_{k\downarrow}\right)$$

where $E_k = \sqrt{\xi_k^2+\Delta^2}$

Lowest energy corresponds to $\langle \gamma^{\dagger}_{k\sigma} \gamma_{k\sigma} \rangle = 0$

$\therefore$ $\gamma^{\dagger}_{k\sigma}$ create quasiparticle excitations (unpaired)

Qu: what is the ground state wave function?

$|\Psi_{GS}\rangle \propto \prod_k \gamma_{k\uparrow} \gamma_{k\downarrow} |0\rangle$ since this guarantees $\gamma_{k\sigma} |\Psi_{GS}\rangle = 0$

vacuum state for original operators

$$= \prod_k v_k (u_k + v_k c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow}) |0\rangle$$

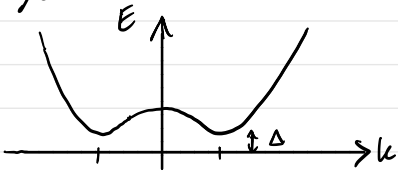
$\therefore$ (normalized) ground state is the BCS wave function:

$$|\Psi_{GS}\rangle = \prod_k (u_k + v_k c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow}) |0\rangle \quad \checkmark$$

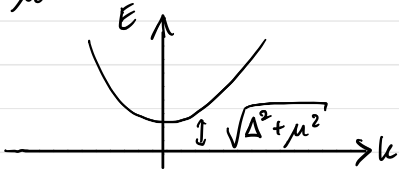
* Gap equation corresponds to $\frac{\partial \langle \hat{H}_{MF} \rangle}{\partial \Delta^2} = 0 \Rightarrow$ minimizes energy

$\rightarrow$ Quasiparticle excitation energy: $E_k = \sqrt{(\varepsilon_k - \mu)^2 + \Delta^2}$

$\mu > 0$



$\mu < 0$



$\rightarrow$ Qualitative change in excitation spectrum @ $\mu = 0$

$\therefore$ $\mu = 0$ marks "crossover point" between BCS and BEC

Within 2D MFT, $\mu = 0 \rightarrow \varepsilon_F = \varepsilon_B/2 \rightarrow \log(k_F a_{2D}) = 0$

#L.3

* Limitations of mean-field theory

- Dimer-dimer interaction is classically scale invariant
  ↳ no quantum anomaly

- No normal state interactions ∴ misses behaviour at weak attraction:

$$\mu = \varepsilon_F - \frac{\varepsilon_F}{\log(k_F a_{2D})} + \dots, \quad \log(k_F a_{2D}) \gg 1$$

- MFT has condensed pairs (i.e., $\Delta \neq 0$) at finite T.

• <u>Pairing at finite temperature</u>

- We have already seen that bosons do not condense at finite T in 2D (for infinite system)
  Similarly, one can also show that $\Delta \neq 0$ only when $T = 0$

- However, interacting system is superfluid for temperatures below a critical temperature T_c

  ↳ Associated with "quasi condensation":

$T < T_c$

condensation in local finite-sized regions

no phase coherence between regions ∴ no global condensation

- In BEC regime, T_c is determined by interactions between dimers (boson–boson interactions)

  $\Rightarrow$ <u>BKT transition</u>

  [Berezinskii, Kosterlitz, Thouless, 1970's]

- In BCS limit, T_c is set by energy required to break pairs
  $\hookrightarrow$ smallest energy scale

- For BCS case, we can estimate T_c using BCS mean-field theory since this captures pair breaking at finite T.
  [Also, system is close to being condensed]

• Mean-field grand potential (for fixed T, μ, Ω):

$$\Phi_{MF} = -\frac{1}{\beta}\log\left(\mathrm{Tr}\left[e^{-\beta \hat{H}_{MF}}\right]\right)$$

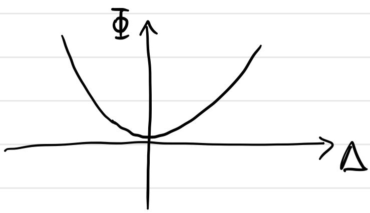
$$= -\frac{\Omega}{g}\Delta^2 + \sum_k \left(\varepsilon_k - \mu - E_k\right) \underbrace{- \frac{2}{\beta}\sum_k \log\left(1 + e^{-\beta E_k}\right)}_{\text{non-interacting gas of quasiparticles}}$$

$T < T_c$

$\rightarrow$ minimum at non-zero Δ

[SF phase]

$T > T_c$



$\rightarrow$ minimum at $\Delta = 0$

[Normal phase]

- Expand Φ close to $\Delta = 0$:

$$\Phi_{MF} = \alpha \Delta^2 + \eta \Delta^4 + \dots$$

[Qu: why even powers?]

Transition corresponds to $\alpha = 0 \Rightarrow \left.\frac{\partial \Phi_{MF}}{\partial \Delta^2}\right|_{\Delta=0} = 0$

$\therefore$ we have:

$$\left. -\frac{\Omega}{g} - \sum_k \frac{1}{2E_k}\left(1 - \frac{2}{1+e^{\beta E_k}}\right)\right|_{\Delta=0} = 0$$

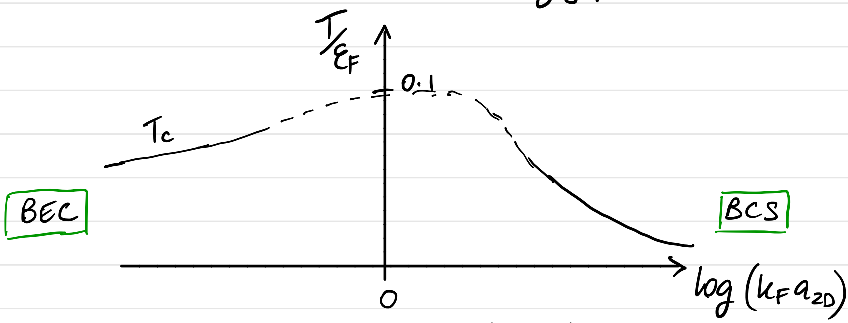
i.e. $$-\frac{\Omega}{g} = \sum_k \frac{1}{2|\xi_k|}\left(1 - \frac{2}{1+e^{\beta|\xi_k|}}\right)$$

$$= \sum_k \frac{1}{2\xi_k}\left(1 - \frac{2}{1+e^{\beta \xi_k}}\right)$$ (check this)

In the limit $\varepsilon_B \ll \varepsilon_F$, this yields:

$$T_c = \frac{e^{\gamma}}{\pi} \underbrace{\sqrt{2\varepsilon_F \varepsilon_B}}_{\text{zero } T \text{ pairing gap}}$$

$\gamma \equiv 0.5772\dots$ (Euler's constant)



- Dimers become non-interacting as $\log(k_F a_{2D}) \longrightarrow -\infty$
  $\therefore T_c \to 0$ (not captured by MFT)
- Maximum T_c around $\log(k_F a_{2D}) \simeq 0$.

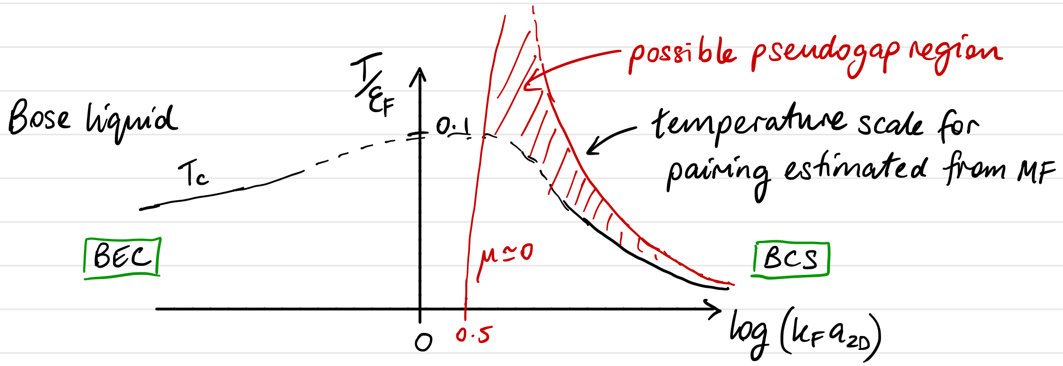
- "Pseudogap" region?

  - Just above T_c we have:
    (1) Bose liquid when $\varepsilon_B/\varepsilon_F \gg 1$, $\log(k_F a_{2D}) \to -\infty$
    (2) Fermi liquid when $\varepsilon_B/\varepsilon_F \ll 1$, $\log(k_F a_{2D}) \to +\infty$
    $\hookrightarrow$ what happens inbetween?

  - The term "pseudogap" comes from the gap-like feature observed in density of states of high-temperature superconductors above T_c

  - Basic question is whether such a pseudogap can be produced by fermion pairing without superconductivity / superfluidity

$\hookrightarrow$ Test with 2D Fermi gas, but to reproduce phenomenology we require a Fermi surface as well as preformed pairs $\Rightarrow \boxed{\mu > 0}$



→ Experimental evidence for such pairing above T_c: 1705.10577

- Remaining question: does pairing above T_c necessarily give a pseudogap in the density of states?

- Equation of state above T_c

  - For $T/T_F \to \infty$, we have $\beta\mu \to -\infty$

    E.g., for non-interacting Fermi gas:

    $$n_k = \frac{1}{1+e^{\beta(\varepsilon_k-\mu)}} \xrightarrow[\beta\mu\to-\infty]{} e^{-\beta(\varepsilon_k-\mu)}$$

    – recover classical Boltzmann gas at high T

∴ in high T limit, $z \equiv e^{\beta\mu} \ll 1$, so we can treat the fugacity z perturbatively

  - Grand potential $\Phi = -\frac{1}{\beta}\log \mathcal{Z}$

    where partition function $\mathcal{Z} = \mathrm{Tr}\left[e^{-\beta(\hat{H}-\mu\hat{N})}\right]$

    Since $\hat{H}$ conserves no. of particles, we can rewrite this as:

    $$\mathcal{Z} = \sum_N \underbrace{e^{\beta\mu N}}_{z^N} \underbrace{\mathrm{Tr}_N\left[e^{-\beta\hat{H}}\right]}_{\substack{\text{trace over states} \\ \text{in N-body cluster}}} \equiv \sum_N z^N B_N$$

∴ we have expansion in fugacity, i.e., the <u>virial expansion</u>:

$$\mathcal{Z} = 1 + zB_1 + z^2B_2 + z^3B_3 + \ldots$$

$$\Rightarrow B_1 = 2\sum_k e^{-\beta\varepsilon_k} = 2L^2\rho\int_0^\infty d\varepsilon\, e^{-\beta\varepsilon} = 2\frac{L^2\rho}{\beta} = \frac{2L^2}{\lambda^2}$$

↑ 2 types of fermions

$\lambda \equiv \sqrt{2\pi/mT}$

$\therefore$ we have $\Phi = -\frac{1}{\beta}\log\left[1+\sum_{N\geq 1} z^N B_N\right]$

$$= -\frac{1}{\beta}\left[\sum_{N\geq 1} z^N B_N - \frac{1}{2}\left(\sum_{N\geq 1} z^N B_N\right)^2 + \frac{1}{3}\left(\sum_{N\geq 1} z^N B_N\right)^3 + \dots\right]$$

$$= -\frac{1}{\beta}\left[B_1 z + \left(B_2 - \frac{1}{2}B_1^2\right)z^2 + \left(B_3 - B_1 B_2 + \frac{1}{3}B_1^3\right)z^3 + \dots\right]$$

– Remarkably, all terms scale as L^2 (higher powers of area cancel)

$\therefore$ the convention is to write it as:

$$\Phi = -\frac{1}{\beta}B_1\left[b_1 z + b_2 z^2 + b_3 z^3 + \dots\right]$$

where b_j are the (dimensionless) virial coefficients.

→ Leading order term (with $b_1 = 1$) is an ideal classical gas

→ Interactions + statistics only enter at order z^2

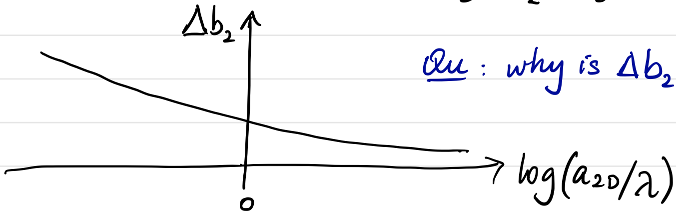
• Total density:

$$n = n_\uparrow + n_\downarrow = -\frac{1}{L^2}\frac{\partial \Phi}{\partial \mu}\bigg|_{T, L^2} = \frac{2}{\lambda^2}\sum_{j\geq 1} j\, b_j\, z^j$$

Non-interacting Fermi gas: $\boxed{b_j^{(0)} = (-1)^{j-1}\frac{1}{j^2}\,,\quad j\geq 1}$

(exercise!)

– Contribution from interactions: $\Delta b_2 = b_2 - b_2^{(0)}$



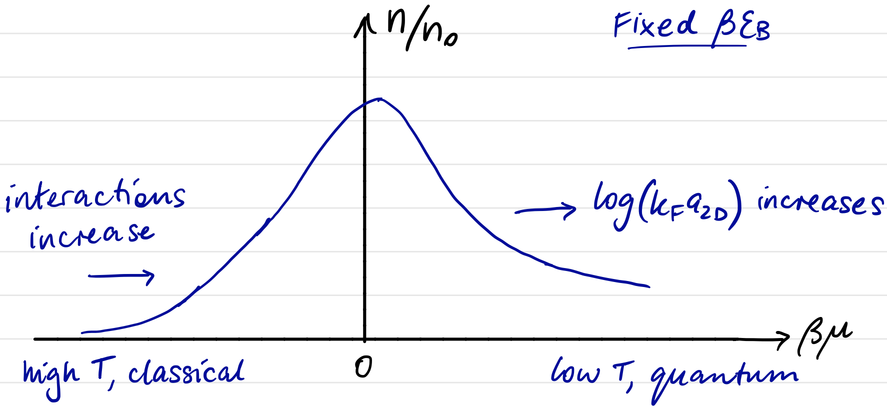
Qu: why is $\Delta b_2 > 0$?

- Density profile in trap $\Rightarrow$ equation of state

  Local density approx. : $\mu(r) = \mu_0 - V(r)$; $r^2 = x^2 + y^2$

$\therefore$ by locally measuring density, we obtain $n(\mu)$ at fixed T, ε_B

- 2D Fermi gas EoS $(T > T_c)$:



$\rightarrow$ n_0 is density of non-interacting Fermi gas at same $\beta\mu$

$\Rightarrow$ Non-monotonic behaviour unlike 3D unitary FG!

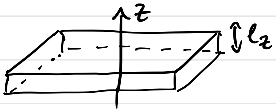
- Crossover from classical to quantum $\Rightarrow$ quantum anomaly
- Recently observed in experiments (Swinburne, Heidelberg)

  [see my Viewpoint, Physics 9, 10 (2016)]

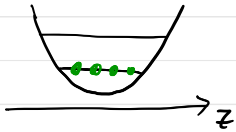
#L4

- The quasi-2D system [Ref: 1408.2737]

- In reality, we live in a 3D world, so we must confine fermions to 2D plane using trapping potential $V_\perp(z)$



Limit of deep potential: $V_\perp(z) \simeq \frac{1}{2} m \omega_z^2 z^2$



- 2D regime: $\varepsilon_F, T \ll \omega_\perp$

OR $1/k_F, \lambda \gg l_z$, where $l_z = \sqrt{\frac{1}{m\omega_z}}$, $\lambda = \sqrt{\frac{2\pi}{mT}}$

- What about interactions?
  - → Real short-range interactions are 3D.
  - → Leads to scattering into higher harmonic levels.

- Consider two-body problem (equal masses)

3D position

$$\sum_{i=1,2}\left[\frac{-1}{2m}\nabla_i^2 + V_\perp(z_i)\right]\Psi(\vec{r}_1,\vec{r}_2) + g_{3D}(\vec{r}_1-\vec{r}_2)\Psi(\vec{r}_1,\vec{r}_2) = E\Psi$$

Now harmonic potential is separable into relative + C.O.M. coordinates (if frequencies are the same):

$$V_\perp(z_1) + V_\perp(z_2) = \frac{1}{2} m_r \omega_z^2 z^2 + \frac{1}{2} M \omega_z^2 z_M^2$$

$\left(m_r = \frac{m}{2}, \quad M = 2m\right)$

∴ we can just consider eq'n in C.O.M. frame:

$$\left[\frac{-1}{2m_r}\nabla_\rho^2 - \frac{1}{2m_r}\frac{d^2}{dz^2} + \frac{1}{2} m_r \omega_z^2 z^2 + g_{3D}(\vec{r})\right]\psi(\vec{\rho}, z) = E\psi(\vec{\rho}, z),$$

where $\vec{\rho} = (x, y)$

Non-interacting eigenstates: $|\vec{k}, n\rangle \equiv e^{i\vec{k}\cdot\vec{\rho}} \underbrace{\phi_n}_{\text{H.O. eigenstates}}(z)$

i.e. non-interacting Hamiltonian:

$$\hat{H}_0 = \sum_{\vec{k},n} \left[\frac{k^2}{2m_r} + (n+1/2)\omega_z\right] |\vec{k}n\rangle\langle\vec{k}n|$$

and interacting part (area set to 1):

$$\hat{H}_{int} = \sum_{\substack{k_1 n_1 \\ k_2 n_2}} \underbrace{\langle k_1 n_1 | \hat{g}_{3D} | k_2 n_2\rangle}_{g_{3D}\phi_{n_1}(0)\phi_{n_2}(0)} |k_1 n_1\rangle\langle k_2 n_2|$$

contact interactions $g_{3D}\delta(r)$

[can also include finite range]

Consider two-body bound state:

$$|\psi_2\rangle = \sum_{k,n} \eta(k,n)\, |k,n\rangle$$

Now we have:

$$(\hat{H}_0 + \hat{H}_{int})|\psi_2\rangle = E\,|\psi_2\rangle$$

$\therefore$ projecting onto $|\vec{k}n\rangle$ gives:

$$\left[\frac{k^2}{2m_r} + (n+\tfrac{1}{2})\omega_z\right]\eta(k,n) + g_{3D}\phi_n(0) \underbrace{\sum_{k',n'} \phi_{n'}(0)\,\eta(k',n')}_{\text{constant } f} = E\,\eta(k,n)$$

i.e., $\eta(k,n) = \dfrac{g_{3D}\phi_n(0)\, f}{E - k^2/2m_r - (n+1/2)\omega_z}$

i.e. $$f = \sum_{k,n} \frac{g_{3D} |\Phi_n(0)|^2 f}{E - k^2/2m_r - (n+1/2)\omega_z}$$

$\therefore$ we finally get:

$$\boxed{\frac{-1}{g_{3D}} = \sum_{k,n} \frac{|\Phi_n(0)|^2}{k^2/2m_r + (n+1/2)\omega_z - E}}$$

[N.B. even n only since $\Phi_n(0) = 0$ when n odd]

But $$\frac{1}{g_{3D}} = \underbrace{\frac{m_r}{2\pi a_s}}_{T_{3D}^{-1}(0)} - \sum_{\vec{k}_{3D}} \frac{1}{k_{3D}^2/2m_r}$$

3D momentum without potential

$\therefore$ cannot neglect sum over n $\rightarrow$ cancels UV divergence

After some clever manipulations, one obtains:

$$\frac{\ell_z}{a_s} = \mathcal{F}(-E/\omega_z + 1/2)$$

where $$\mathcal{F}(x) = \int_0^\infty du \frac{1}{\sqrt{4\pi u^3}} \left[1 - \frac{e^{-xu}}{\sqrt{(1-e^{-2u})/2u}}\right]$$

For $|x| \ll 1$:

$$\mathcal{F}(x) \simeq \frac{1}{\sqrt{2\pi}} \log(\pi x/B) + \frac{\log(2)}{\sqrt{2\pi}} x + \cdots, \quad B \simeq 0.905$$

$\Rightarrow$ Always a solution for any 3D scattering length a_s

i.e. there is always a bound state in quasi-2D, even when there are none in 3D.

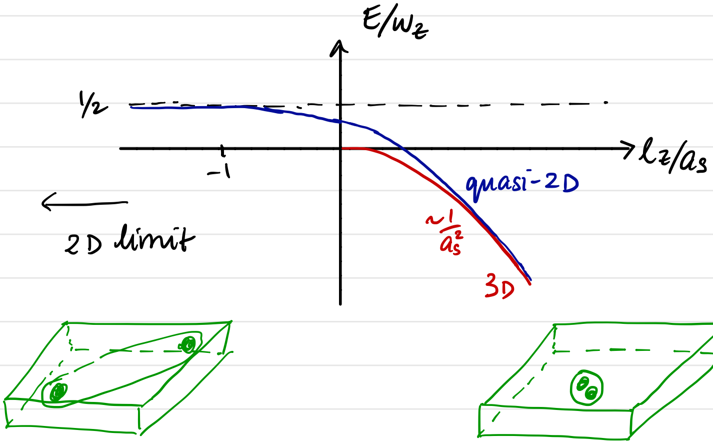
[Qu: how is this possible?]

Keeping lowest order (2D) term in $\mathcal{F}(x)$, we have binding energy:

$$\varepsilon_B \equiv -E + \frac{1}{2}\omega_z \simeq \frac{\omega_z B}{\pi} \exp\left(\sqrt{2\pi}\, l_z/a_s\right)$$

$\rightarrow$ corresponds to limit $l_z/a_s \ll -1$.

- Two-body energy across interaction range:



- The quasi-2D T matrix is:

$$\mathcal{T}(E) = \frac{\sqrt{2\pi}}{m_r}\left[\frac{l_z}{a} - \mathcal{F}\left(-E/\omega_z + 1/2\right)\right]^{-1}$$

$\Rightarrow$ when $\left|-E/\omega_z + 1/2\right| \ll 1$, we recover 2D expression:

$$\mathcal{T}(E) \simeq \frac{2\pi}{m_r}\left[\log\left(\frac{1}{2m_r a_{2D}^2 E}\right) + i\pi\right]$$

where

$$\boxed{a_{2D} = l_z\sqrt{\frac{\pi}{B}}\exp\left(-\sqrt{\frac{\pi}{2}}\frac{l_z}{a_s}\right)}$$

Petrov + S, 2001

- Points to note:
  - We require $l_z/a_s \ll -1$ for bound state to be 2D, with $\varepsilon_B \simeq \frac{1}{2m_r a_{2D}^2}$
  - We require collision energy $\varepsilon \equiv E - \frac{1}{2}\omega_z \ll 1$ to have 2D scattering states → relevant parameter is a_{2D} not ε_B!
  - When $|a_s| \ll l_z$, there is a large range of energies such that:

$$\mathcal{T}(\varepsilon) \simeq \frac{\sqrt{2\pi}}{m_r}\frac{a_s}{l_z}$$

→ independent of ε, recovers scale invariance!

//

OUTLOOK:

- What interesting physics lies in the crossover between 2D and 3D?

e.g., is T_c for superfluidity maximal inbetween?