Tuesday, 2 October 2018 08:12

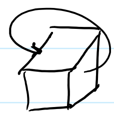
Quantum Physics in 1D

0.] Why 1D:

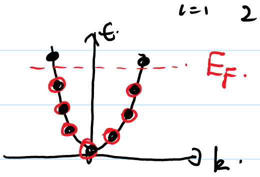
I.] Reminder of "high dimensional" fermions.

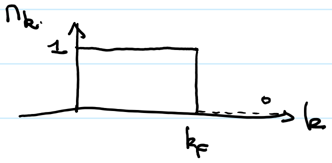
1.) Free fermions.

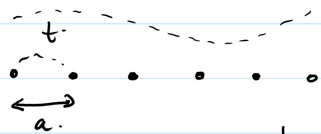
$$H = \sum_{i=1}^{N} \frac{P_i^2}{2M}. \qquad \epsilon(k) = \frac{k^2}{2m}.$$



$$k_x = \frac{2\pi}{L_x} n_x$$





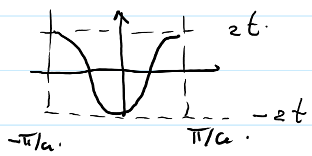


$$H = -t \sum_i |i+1\rangle\langle i| \qquad \text{tight binding model.}$$

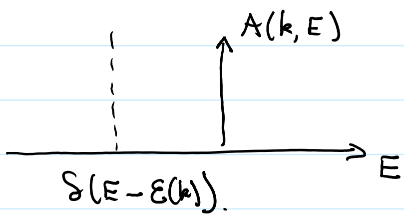
$$|k\rangle = \sum_j e^{ikr_j} |j\rangle. \qquad H = \sum_k \epsilon_k |k\rangle\langle k|$$

$$\epsilon_k = -2t\cos(ka)$$

$$k \in \left[-\frac{\pi}{a}, \frac{\pi}{a}\right]$$



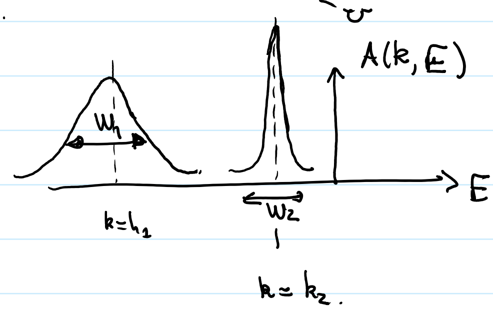
$A(k, E)$.



2.) Fermi Liquids 101:

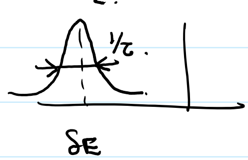
interaction U naive answer

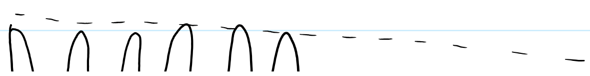
Landau Fermi liquid theory.

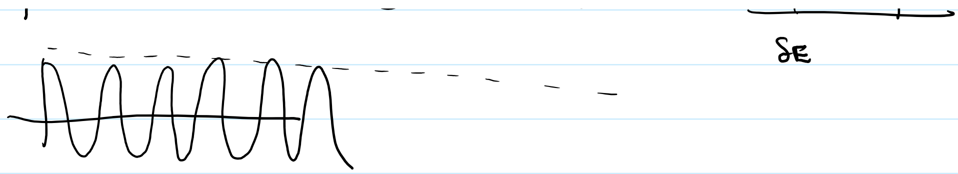


$$W \ll \delta E = E - E_F.$$

$$\psi(t) = e^{i\delta E t}\, e^{-|t|/\tau}. \quad \xrightarrow{FT}$$



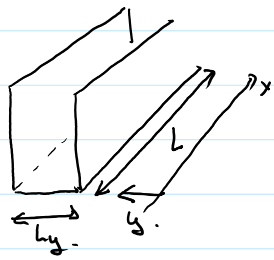




δE

II] 1D Quantum systems

1.) What does 1D mean?



$$\frac{P_x^2}{2m} + \frac{P_y^2}{2m}.$$

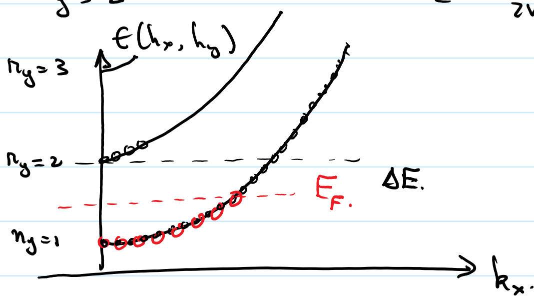
$$\epsilon(k_x, k_y) = \frac{k_x^2}{2m} + \frac{k_y^2}{2m}.$$

$$k_x = \frac{\pi}{L_x} n_x \qquad k_y = \frac{\pi}{L_y} n_y \qquad \sin(k_y y)$$

L_y small $\quad$ L_x very large.

$$n_y = 1 \qquad \epsilon(k_x, k_y) = \frac{1}{2m}\left[\left(\frac{\pi}{L_y}\right)^2 + k_x^2\right]$$

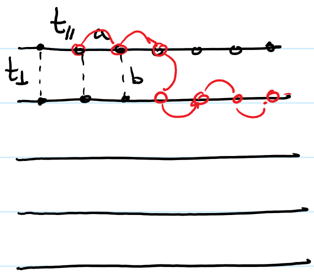
$$n_y = 2 \qquad\qquad = \frac{1}{2m}\left[\frac{4\pi^2}{L_y^2} + k_x^2\right]$$



$$\vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4.$$

$U \ll \Delta E.$

cannot change the quantum number n_y.



$$\epsilon = -2t_{\parallel}\cos(k_{\parallel} a) - 2t_{\perp}\cos(k_{\perp} b)$$

$t_{\parallel} \gg t_{\perp}$. $\qquad$ $t_{\parallel} \gg T \gg t_{\perp}$.

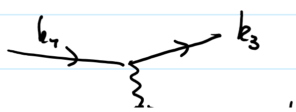
$\hookrightarrow$ effectively 1D.

2.) Interacting bosons:

$$H = \sum_{j=1}^{N} \frac{P_j^2}{2m} + g \sum_{\langle ij \rangle} \delta(R_i - R_j)$$

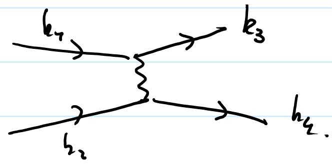
Lieb-Liniger model.

$$H = \sum_k \underbrace{\epsilon_k}_{\frac{k^2}{2m}} b_k^+ b_k + \frac{g}{\Omega} \sum_{\substack{k_1 k_2 \\ k_3 k_4}} \delta_{k_1+k_2, k_3+k_4} \, b_{k_3}^+ b_{k_4}^+ b_{k_2} b_{k_1}$$



$[b_k, b_{k'}^+] = \delta_{k,k'}$

$\overline{2m}$.

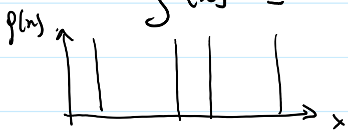


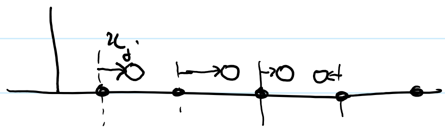
$$[\, b_{k_1}, b^{+}_{k_2}] = \delta_{k_1 k_2}.$$
$$[\, b_{k_1}, b_{k_2}] = 0$$

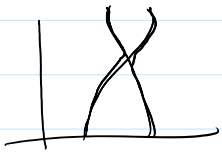
$$\langle b^{+} \rangle$$

Duncan Haldane :

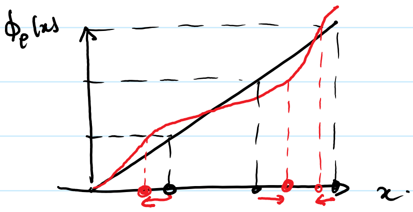
$$\rho(x) = \sum_j \delta(x - x_j)$$







$$\phi_\rho(x_j) = 2\pi n. \qquad n \text{ : integer.}$$



$$\rho(x) = \sum_p \frac{\delta(\phi_\rho(x) - 2\pi p)}{|\nabla_x \phi_\rho(x)|}.$$

$$\rho(x) = \left[\rho_0 - \frac{1}{\pi}\nabla\phi\right] \sum_{p \text{ integer}} e^{i 2p(\pi\rho_0 x - 2\phi(x))}.$$

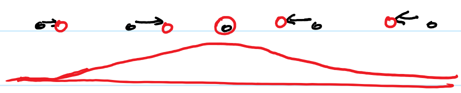
$$\phi_\rho(x) = 2\pi\rho_0 x - 2\phi(x)$$

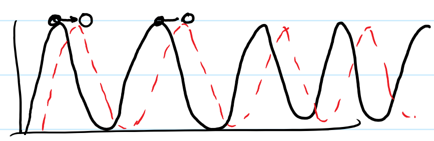
$\underline{p=0}$

$$\rho(x) = \rho_0 - \frac{1}{\pi}\nabla\phi$$

$p = 1, -1$

$$\rho(x) = \rho_0 \cos(2\pi\rho_0 x - \underbrace{2\phi(x)}_{})$$





$$b_x = [\rho(x)]^{1/2} e^{i\theta(x)}.$$

$$\phi(x), \ \theta(x).$$
$$H[\phi(x), \theta(x)] = \ ?$$

$$[\, b_x, b^{+}_{x'}] = \delta(x - x') \quad \Leftarrow \quad [\phi(x), \underbrace{\tfrac{1}{\pi}\nabla\theta(x')}_{\Pi_\phi(x).}] = i\,\delta(x - x').$$

Rewrite H using $\phi(x)$ $\theta(x)$. (or $\Pi_\phi(x)$)

Kinetic. $\displaystyle\sum_j \frac{P_i^2}{2M}$ $\qquad b^{+}(x)$. $\qquad H = \int dx \frac{1}{2m}(\nabla b_x)^{+}(\nabla b_x)$

$P|\psi\rangle = \nabla \cdot \psi$ $\qquad b = [\rho(x)]^{1/2} e^{i\theta(x)}$

$P(\phi) = \nabla \cdot \psi \qquad b = [\rho(x)]^{1/2} e^{i\theta(x)}$

$$\nabla b = \nabla \rho^{1/2} e^{i\theta} + \rho^{1/2} e^{i\theta} i \nabla\theta.$$
$$\nabla b^+ = \nabla \rho^{1/2} e^{-i\theta} - \rho^{1/2} e^{-i\theta} i \nabla\theta.$$

$$(\nabla b)^+ (\nabla b) = (\nabla \rho^{1/2})^2 + \rho (\nabla\theta)^2.$$
$$= \left(\frac{1}{2\rho^{1/2}} \nabla\rho\right)^2 + \rho(x) (\nabla\theta_x)^2.$$
$$= \frac{1}{4\rho} (\nabla\rho)^2 + \rho(x) (\nabla\theta)^2.$$

$$H_{kin} = \int dx \left[\frac{1}{4\rho} (\nabla\rho)^2 + \rho(x) (\nabla\theta)^2 \right]$$

$\rho(x) \sim \rho_0 - \frac{1}{\pi} \nabla\phi$ + oscillating.

$\nabla\rho \sim -\frac{1}{\pi} \nabla^2\phi.$

$$\approx \int dx \left[\frac{1}{\pi^2 4 \rho_0} (\nabla^2\phi)^2 + \rho_0 \underbrace{(\nabla\theta)^2}_{(\pi\Pi_\phi)^2} \right] \quad + \,?\, \nabla\phi(\nabla\theta)^2 + \text{oscillating}$$

$$= \sum_k \left[\frac{1}{4\pi^2\rho_0} k^4 \phi_k^* \phi_k + \rho_0 \pi^2 \Pi_k^* \Pi_k. \right]$$

$$H_{int.} = \frac{g}{2} \int dx \int dx' \underset{\delta(x-x')}{V(x-x')} \underset{(\rho(x)-\rho_0)(\rho(x')-\rho_0)}{\rho(x)\rho(x')} \equiv \frac{g}{2} \int dx \, (\delta\rho(x))^2.$$

$\rho \sim \rho_0 - \frac{1}{\pi} \nabla\phi$ + oscillations

$$H_{int} = \frac{g}{2} \int dx \, \frac{1}{\pi} (\nabla\phi)^2.$$

$$H = \int dx \left[\frac{\rho_0 \pi^2}{2m.} (\Pi_\phi(x))^2 + \frac{g}{2\pi} (\nabla\phi)^2 + \frac{1}{4\pi^2\rho_0} (\nabla^2\phi)^2 \right]$$

$$H = \sum_k^{\Lambda \sim 1/a.} \left\{ \frac{\rho_0 \pi^2}{2m.} \Pi_k^* \Pi_k + \left[\frac{g}{2\pi} k^2 + \frac{k^4}{4\pi^2\rho_0} \right] \phi_k^* \phi_k. \right\}$$

$$E^2 \sim \frac{1}{\rho_0 \pi^2} \left[\frac{g k^2}{2\pi} + \frac{k^4}{4\pi^2 \rho_0} \right]$$

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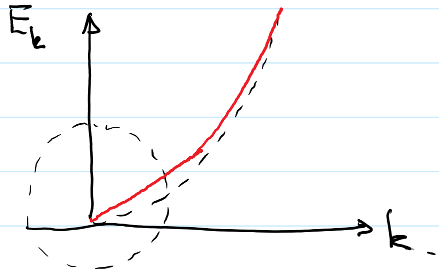


$$H = \frac{1}{2\pi}\int dx \left[uK\,(\pi \Pi_\phi)^2 + \frac{u}{K}(\nabla\phi)^2\right]$$

$K \propto \frac{1}{\sqrt{g}}$.

$$\begin{cases} uK = \rho_0/2m. \\ \frac{u}{K} = \frac{g}{2\pi}. \end{cases}$$

u : dimensions of a velocity.

K dimensionless.

$E = u\,k$.

$$\begin{cases} b(x) = [\rho(x)]^{1/2}\, e^{i\Theta(x)} \\ \rho(x) = \left(\rho_0 - \frac{1}{\pi}\nabla\phi\right)\sum_p e^{i2p(\pi\rho_0 x - 2\phi(x))} \end{cases}$$

$$\langle b(x)\rangle \approx \rho_0^{1/2}\,\langle e^{i\Theta(x)}\rangle \qquad \Theta(x) \leftarrow \sum_k a_k + a_k^\dagger.$$

$$\langle e^{\sum a + a^\dagger}\rangle = e^{+\frac{1}{2}\langle(\ )^2\rangle}$$

$$\int e^{-ax^2+by} \rightarrow e^{-b^2(\ )}$$

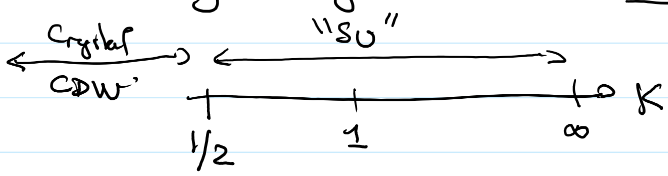
$$= 0 \qquad e^{-\frac{1}{2}\langle\Theta^2(x)\rangle}$$

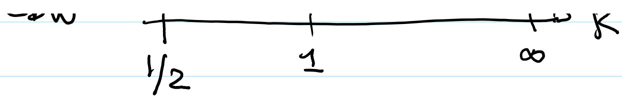
$$\begin{cases} \langle b(x)\, b^\dagger(0)\rangle \propto A\left(\frac{1}{x}\right)^{1/2K}. \\ \langle \delta\rho(x)\,\delta\rho(0)\rangle \sim B\,\frac{1}{x^2} + C\cos(2\pi\rho_0 x)\left(\frac{1}{x}\right)^{2K} \end{cases}$$

$$+ D\cos(4\pi\rho_0 x)\left(\frac{1}{x}\right)^{8K}.$$

$$+ \cdots$$

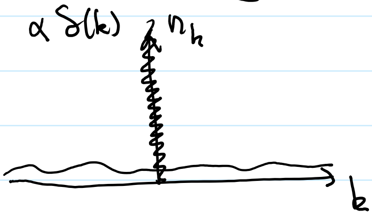
Quasi long range order both $\begin{cases}\text{Superfluid.} \\ \text{density (Crystal).}\end{cases}$



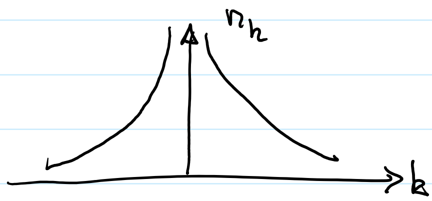


$$n(k) = \int dx \; e^{ikx} \langle b(x)\, b^\dagger(0) \rangle \sim k^{\left[1/2K - 1\right]}$$

$$(1/x)^{1/2K}$$



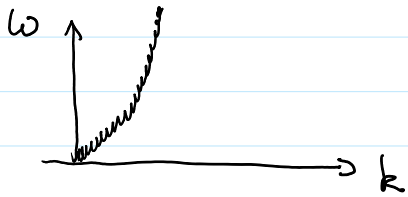
BEC



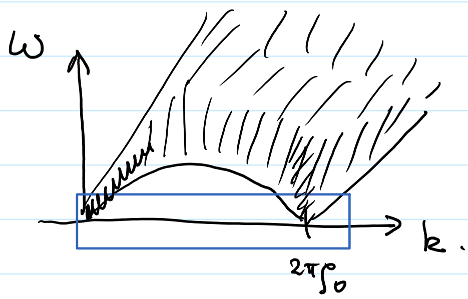
1D bosons.

$$\langle \delta\rho(x,t)\, \delta\rho(0,0) \rangle$$

$\rightarrow \; S(k,\omega)$



BEC
+ Bogoliubov.



$\rightarrow$ Spins ; Fermions.

$$[b(x), b^\dagger(x')] = \delta(x-x') \qquad \{c(x), c^\dagger(x')\} = \delta(x-x').$$

$$c(x) = b(x)\, e^{i\frac{1}{2}\phi_\rho(x)}.$$

$\rho_c \rightarrow$ same decomposition.

$$\phi_\rho(x) = 2\pi\rho_0 x - \underbrace{2\phi(x)}_{\frac{2}{\pi}\int_{-\infty}^{x} dy\, \rho(y).}$$

$$\rho(x) \sim -\frac{1}{\pi}\nabla\phi.$$

III] Concept of Tomonaga-Luttinger liquid.

1) Universality of the solution:

$$\begin{cases} H = \dfrac{1}{2\pi}\displaystyle\int dx \;\; uK\,(\pi\Pi_\phi)^2 + \dfrac{u}{K}(\nabla\phi)^2 \\ b = [\rho]^{1/2} e^{+i\theta} \\ \rho = [\rho_0 \quad\quad]\end{cases}$$

$$\begin{cases} \theta = (\,\beta_0\,) \\ \rho = (\rho_0 - \quad) \sum \end{cases}$$

exactly (up to amplitudes) the asymptotic (large space / time) correlations.

provided that one uses the exact values of u and K.

- g small : perturbation is a good approximation.
- Extract them where you can: → exact solution
  → Smart ideas.
  → Numerics.

$K \sim \frac{u}{K}$. $\quad C_v \sim \frac{T}{u}$. $\quad \frac{E(L) - E(\infty)}{L} \sim \frac{u}{L^2}$.